ALGEBRAS OF FUNCTIONS WITH FOURIER TRANSFORMS IN L_p

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CERTIFICATE

This is to certify that the work embodied in the thesis "Algebras of functions with Fourier transforms in L_p" by Ashok Kumar Gupta has been carried out under my supervision and has not been submitted elsewhere for a degree.

August - 1975

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CONTENTS

CHAPTER - I

INTRODUCTION

The work of Larsen, Liu and Wang [20] on algebras $A_p(G)$ $(1 \leq p < \infty)$ of integrable functions on (locally compact abelian group) G with Fourier transforms in $L_p(\Gamma)$ stimulated many mathematicians to investigate these algebras and their generalisations further, namely, Figa-Talamanca and Gaudry [9] , Lai [21], [22] , Larsen [16-20] , Martin and Yap [25] , Reiter [27] , Yap [32-35] , Wang [31] and Burnham [1-3] . Most of the study has been to compare these algebras with group algebra $L_1(G)$. These algebras are similar to group algebra $L_1(G)$ in many ways. Authors of [20] showed that these algebras are commutative semi-simple Banach algebras and that their maximal ideal space can be identified with Γ. The characterisation of multipliers of $A_p(G)$'s as $M(G)$ for noncompact G was asserted in [20] . Figa-Talamanca and Gaudry prove the fact in [9] .

It is well known that factorisation holds in group algebra $L_1(G)$, that is, $L_1(G) * L_1(G) = L_1(G)$. Martin and Yap [25] showed that factorisation does not hold in $A_p(G)$. So property of factorisation distinguishes these algebras from the group algebra.

The ideal structure of $A_p(G)$ algebras has been investigated in several papers. Larsen, Liu and Wang [20] showed that there

exists a one to one correspondence between closed ideals of $A_p(G)$ and $L_1(G)$, while Lai showed that such a correspondence exists between closed primary ideals of $A_p(G)$ and $L_1(G)$. In Martin and Yap [25] , however, it was proved that there exist maximal ideals in $A_p(G)$ which are neither closed, primary nor regular, so the ideal structure of $A_p(G)$ and $L_1(G)$ are not completely identical.

$A_p(G)$ algebras are part of a wider class of algebras known as Segal algebras; see Reiter [27] . Another generalisation of these algebras is given by Yap [34] .

Our interest in these algebras arose as a result of an attempt to prove the distinctness of $A_p(G)$'s for distinct p's, in case G is nondiscrete. Later on, works of Figa-Talamanca and Gaudry [9],[10] , Price [26] and Saeki [29] concerning some problems on L_p-multipliers motivated us to solve some corresponding problems for A_p-multipliers.

The present thesis consists of five chapters.

In the second chapter we set our notations and state preliminary results. Basic definitions and well known results which are needed later are also accumulated.

In Chapter III we show that if $1 \leq p < q < \infty$, then

(i) $\bigcup_{1 \leq p < q} A_p(G)$ is a dense linear subspace of category I in $A_q(G)$.

(ii) The ideal generated by $A_p(G) * A_q(G)$ is properly contained in $A_p(G)$.

(iii) The ideal generated by $S(G) * L_p(G)$ is properly contained in $L_p(G)$ provided G is infinite compact abelian and $S(G)^\wedge \subseteq L_{p_0}(\Gamma)$ for some $p_0, 0 < p_0 < \infty$.

These results answer the questions raised by Larsen in 'B' and 'K' of [18] . The result mentioned in (ii) is proved in a greater generality. We accomplish (i) by using structure of a nondiscrete locally compact abelian group and a form of open mapping theorem. And (ii) is accomplished by using an idea due to Burnham [1] and Wang [31] .

We have divided Chapter IV into three sections. In the first section we study analogues of the following results about L_p-multipliers. Figa-Talamanca and Gaudry [9] have proved that $M_{L_p}(\Gamma) \cap c_0(\Gamma) \neq M_{L_q}(\Gamma) \cap c_0(\Gamma)$ provided $1 \leq p < q \leq 2$ and G is infinite. Price [26] has proved even stronger results, namely

$$\bigcup_{1 \leq q < p} M(L_q(G)) \subsetneq M(L_p(G)) \subsetneq \bigcap_{p < q \leq 2} M(L_q(G))$$

for $1 < p < 2$, with the first inclusion remaining strict when $p = 2$ and second for $p = 1$ (similar results also hold for $2 \leq p \leq \infty$). To look for the analogues of these results for $A_p(G)$ algebras, we see that $M_{A_p}(\Gamma) = M(G)^\wedge$ provided G is

noncompact abelian and $M_{A_p}(\Gamma) = C(\Gamma)$ provided G is infinite compact abelian and $1 \leq p \leq 2$. Therefore problem is interesting only when $2 < p < \infty$ and G infinite compact abelian. In Section 1 of Chapter IV it is shown that if $2 < q < p < \infty$, then

(i) $M_{A_p}(\Gamma) \cap c_0(\Gamma) \subsetneqq M_{A_q}(\Gamma) \cap c_0(\Gamma)$

(ii) $\bigcup_{2<q<p} M_{A_p}(\Gamma)$ is of first category in $M_{A_q}(\Gamma)$.

In Section 2 of Chapter IV we study $M_{A_p}^{A_q}(\Gamma)$ for $1 \leq p,q < \infty$. For infinite compact abelian groups G relationship between $M_{A_p}^{A_q}(\Gamma)$ and $\ell_r(\Gamma)$, $1 \leq r \leq \infty$ is obtained. The following are the main results of this section.

(i) $M_{A_p}^{A_q}(\Gamma) = \ell_{pq/p-q}(\Gamma) \quad 1 \leq q < p \leq 2$

(ii) $\ell_{pq/p-q}(\Gamma) \subsetneqq M_{A_p}^{A_q}(\Gamma) \subsetneqq \ell_{2q/2-q}(\Gamma)$ if $1 \leq q \leq 2 < p$.

Moreover, $\ell_{pq/p-q}(\Gamma)$ is the largest $\ell_r(\Gamma)$ space contained in $M_{A_p}^{A_q}(\Gamma)$.

(iii) For some values of p,q satisfying $2 < q < p < \infty$, $\ell_{pq/p-q}(\Gamma)$ need not be contained in $M_{A_p}^{A_q}(\Gamma)$.

In Section 3 of Chapter IV we study an analogue of the following results about L_p-multipliers. Saeki [29] shows that

if Γ_o is a closed subgroup of Γ and $\phi \in M_{L_p}(\Gamma) \cap C(\Gamma)$, then $\phi|_{\Gamma_o} \in M_{L_p}(\Gamma_o) \cap C(\Gamma_o)$. Conversely it is proved by Figa-Talamanca and Gaudry [10] that, if $p = 1$ or 2 or Γ_o is a closed discrete subgroup of Γ, then, every function in $M_{L_p}(\Gamma_o) \cap C(\Gamma_o)$ is the restriction of some function in $(M_{L_p}(\Gamma)) \cap C(\Gamma)$. Since functions in $M_{A_p}(\Gamma)$ are continuous, we have only to consider $M_{A_p}(\Gamma)$ instead of $M_{A_p}(\Gamma) \cap C(\Gamma)$. In this section we show the following for $1 \leq p,q < \infty$.

(i) If G is noncompact abelian, then $M_{A_p}(\Gamma)|_{\Gamma_o} = M_{A_p}(\Gamma_o)$, unless Γ_o is a closed discrete subgroup of Γ, in which case $M_{A_p}(\Gamma)|_{\Gamma_o} \subsetneqq M_{A_p}(\Gamma_o)$

(ii) If G is an infinite compact abelian group, then

$$M_{A_p}^{A_q}(\Gamma)|_{\Gamma_o} = M_{A_p}^{A_q}(\Gamma_o).$$

In Chapter V we study the class $E_p(G)$ of continuous functions with p-summable Fourier transforms on infinite compact abelian groups G. Since Fourier transform of every function in $C(G)$ is in $\ell_2(\Gamma)$, therefore we may assume that $1 \leq p \leq 2$. For $G = T$, this class of functions has been studied by ~~Lynette~~ Butler-Bloom [23], [24]. In this chapter we show that $E_p(G)$ is a Segal algebra. Furthermore, results proved in Chapters III and IV about $A_p(G)$ algebras are also proved for $E_p(G)$ algebras. Besides this, we have given some results

concerning multipliers between $A_p(G)$ and $E_q(G)$ algebras $1 \leq p < \infty$, $1 \leq q \leq 2$. The main results are

(i) $M_{A_p}^{E_q}(\Gamma) = \ell_{p'}(\Gamma) \quad 1 \leq p \leq 2$

(ii) $\ell_{p'}(\Gamma) \subsetneq M_{A_p}^{E_q}(\Gamma) \subsetneq (J_p(G))^{\wedge}$, $2 < p$

(We have assumed $q > 1$ for proving the first proper inclusion in (ii)).

Finally, at appropriate places it is noted that most of the results proved in this thesis are valid for arbitrary infinite compact groups.

CHAPTER - II

PRELIMINARIES AND BASIC RESULTS

2.1 NOTATIONS : All the groups which we consider will be Hausdorff topological groups. These need not be abelian unless explicitly stated. Abelian groups will be written additively. Every linear space considered will be over $\mathbb{C}$, the field of complex numbers.

Let G be a locally compact group. Then we shall use following notations.

λ for the left Haar measure on G .

For $0 < p < \infty$, $L_p(G)$ is the space of all measurable functions on G whose pth powers are integrable with respect to λ . If $1 \leq p < \infty$, then $L_p(G)$ is a Banach space with norm

$$||f||_p = \left(\int_G |f|^p \, d\lambda \right)^{1/p}$$

With **convolution** as multiplication $L_1(G)$ is a Banach algebra.

$L_\infty(G)$ stands for the Banach space of all measurable functions on G which are essentially bounded with respect to λ , the norm of an element $f \in L_\infty(G)$ being

$$||f||_\infty = \operatorname*{ess\,sup}_{x \in G} |f(x)|$$

For an index $p(1 \leq p \leq \infty)$, p' is the index satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

The space of all bounded complex regular measures on G is denoted by $M(G)$. It is a Banach algebra with convolution as product and norm

$$||\mu|| = |\mu|\ (G),$$

$|\mu|$ being the total variation of μ.

We write $C_o(G)$ for the Banach space of all continuous complex valued functions on G vanishing at infinity with norm

$$||f||_\infty = \sup_{x \in G} |f(x)|$$

$C_c(G)$ is the subspace of $C_o(G)$ consisting of those functions having compact support. If G is compact then we shall write $C_c(G) = C_o(G) = C(G)$.

The translations $_xf$ and f_x of a function f on G are defined by $_xf(y) = f(x^{-1}y)$ and $f_x(y) = f(yx^{-1})$ for every $y \in G$.

A Segal algebra on G (see definition 2.2.1) is denoted by $S(G)$.

For a closed (normal) subgroup H of G, Π_H is the mapping from $C_c(G)$ into $C_c(G/H)$ defined by

$$\Pi_H(f)\ (xH) = \int_H f(xy)\ dy$$

where dy denotes the Haar integral on H.

The dual group of a locally compact abelian group G is denoted by Γ and η stands for the Haar measure on Γ. When Γ is discrete, we write $\ell_p(\Gamma)$ for $L_p(\Gamma)$. Let $C(\Gamma)$ denote the Banach algebra of all bounded continuous complex valued functions on Γ with pointwise operations and supremum norm.

Σ denotes the dual object of an infinite compact group (see [12, 27.3]). For $P \subseteq \Sigma$, $[P]$ is as in [12,27.35]. For each $\sigma \in \Sigma$, select a fixed member $U^{(\sigma)}$ of σ with representation space H_σ. Let d_σ be the (finite) dimension of H_σ, $\beta(H_\sigma)$ the space of all bounded linear operators on H_σ and I_{d_σ} the identity operator on H_σ.

$C(\Sigma_0)$ $(\Sigma_0 \subseteq \Sigma)$ stands for the cartesian product $\underset{\sigma\in\Sigma_0}{P} \beta(H_\sigma)$; it is a $\sim$ algebra with scalar multiplication, addition, multiplication and adjoint of an element defined coordinatewise.

We write $C_p(\Sigma_0)$ $(1 \le p \le \infty)$ for the space of those elements $E = (E_\sigma)$ in $C(\Sigma_0)$ for which

$$||E||_p = \Big(\sum_{\sigma\in\Sigma_0} d_\sigma ||E_\sigma||^p_{\phi_p}\Big)^{1/p} < \infty \text{ if } 1 \le p < \infty$$

$$||E||_\infty = \sup \{||E_\sigma||_{\phi_\infty} : \sigma \in \Sigma_0\} < \infty$$

where the operator norms $||\ ||_{\phi_p}$ are as defined in [12,(D.37),(D.36e)].

'$\wedge$' denotes the Fourier or the Fourier-Stieltjes transform. '$\vee$' stands for the inverse Fourier or Fourier-Stieltjes transform.

For the definition of algebras $A_p(G)$, see (2.2.2(iv)).

The subalgebra $\{\mu \in M(G) \mid \hat{\mu} \in L_p(\Gamma)\}$ of $M(G)$ for a locally compact abelian group G is denoted by $B_p(G)$. For an arbitrary compact group G, the same definition holds with $C_p(\Sigma)$ in place of $L_p(\Gamma)$. If G is locally compact abelian $B(\Gamma) = \{\hat{\mu} : \mu \in M(G)\}$

For the definitions of multiplier spaces $M(X,Y)$, $M_X^Y(\Gamma)$, $M_X^Y(\Sigma)$, $M(X)$, $M_X(\Gamma)$, $M_X(\Sigma)$, $[\ell_p, \ell_q]$ and the norm $||\,||_S$ of an element of $M_S(\Gamma)$ see 2.3, 2.3.4 and 2.4.1.

Finally for any set E, χ_E denotes the characteristic function of E.

2.2 SEGAL ALGEBRAS

DEFINITION 2.2.1 : Let G be an arbitrary locally compact group. A linear subspace of $L_1(G)$ is said to be a Segal algebra, and denoted by $S(G)$, if it satisfies the following conditions.

(i) $S(G)$ is dense in $L_1(G)$.

(ii) $S(G)$ is a Banach space under some norm $|| \; ||_S$,

(2.1.1a) $$||f||_S \geq ||f||_1 \quad \text{for all } f \in S(G).$$

(iii) $S(G)$ is left invariant ($f \in S(G) \Rightarrow {}_yf \in S(G)$ for all $y \in G$) and for each $f \in S(G)$ the mapping $y \to {}_yf$ of G into $S(G)$ is continuous.

(iv) The Segal norm is left invariant : $||{}_yf||_S = ||f||_S$ for all $f \in S(G)$ and $y \in G$.

(ii) says only that $S(G)$ is a Banach space, but it can be easily shown that $S(G)$ is, in fact, a Banach algebra under convolution. Also, it would be enough to require in place of (2.1.1a) the existence of a constant C such that

$$||f||_1 \leq C \, ||f||_S \quad \text{for all } f \in S(G) \; ;$$

but the assumption $C = 1$ is no loss of generality: if $C > 1$, w can replace the given norm $||f||_S$ by the equivalent norm $C \; ||f$

(this even works for $S(G)$ as a Banach algebra). On the other hand, if G is abelian, then (ii) can be replaced by the condition that $S(G)$ be a Banach algebra under the norm $|| \ ||_S$.

2.2.2 EXAMPLES OF SEGAL ALGEBRAS

(i) $L_1(G)$ itself is a Segal algebra .

(ii) The continuous functions in $L_1(G)$ that vanish at ∞ form a Segal algebra, the norm being defined by $||f||_S = ||f||_1 + ||f||_\infty$.

(iii) For $1 < p < \infty$ the intersection $L_1(G) \cap L_p(G)$ is a Segal algebra, with norm $||f||_S = ||f||_1 + ||f||_p$.

(iv) Let G be a locally compact abelian or an infinite compact group and $1 \leq p < \infty$. Then $A_p(G) = \{f \in L_1(G) | \hat{f} \in L_p(\Gamma)$ (or $C_p(\Sigma))\}$ is a Segal algebra with norm $||f||_{A_p} = ||f||_1 + ||\hat{f}||_p$.

(v) Let $G = R$ (the group of real numbers). The functions $f \in L_1(R)$ which are absolutely continuous and such that f' (the derivative of f) is integrable over R form a Segal algebra under the norm $||f||_S = ||f||_1 + ||f'||_1$.

2.2.3 PROPERTIES OF SEGAL ALGEBRAS :

(i) Any Segal algebra $S(G)$ is a left ideal of $L_1(G)$ and

$$||h * f||_S \leq ||h||_1 \, ||f||_S \quad f \in S(G), \ h \in L_1(G).$$

In particular, $S(G)$ is a Banach algebra under $|| \ ||_S$.

(ii) Let μ be a bounded, complex measure on G. Then $\mu * f$ is in $S(G)$ for any $f \in S(G)$ and

$$||\mu * f||_S \leq ||\mu|| \; ||f||_S.$$

(iii) Every Segal algebra has multiple approximate left identity that have L_1-norm 1. If G is abelian, then every Segal algebra on G has multiple approximate units of L_1-norm 1 and having a positive Fourier transform with compact support.

The following theorems contain basic facts about Segal algebras on locally compact abelian groups. We state these without proofs ; for proofs we indicate the relevant references.

THEOREM 2.2.4 : Every Segal algebra satisfies Wiener-Ditkin's condition.

PROOF : See Yap [33] . The proof is also implicitly contained in the book of Hewitt and Ross [12] .

THEOREM ^.2.5 : Every Segal algebra is semisimple, regular and Tauberian.

PROOF : For semisimplicity see Yap [33] , and for the other properties see Reiter [27] .

THEOREM 2.2.6 : Every closed ideal I_S of $S(G)$ is of the form $I_S = I \cap S(G)$ for a unique closed ideal I in $L_1(G)$. In fact, I is the closure of I_S in $L_1(G)$.

PROOF : See Reiter [27]

2.3 MULTIPLIERS BETWEEN SPACES : Let X and Y be translation invariant topological linear spaces of functions or measures defined on a locally compact abelian group G for which it is possible to define Fourier or Fourier-Stieltjes transforms. A continuous linear transformation T from X into Y is caled a multiplier if T commutes with translations. Let T be a linear transformation from X to Y. Suppose there exists a function ϕ on Γ such that $(Tf)^{\wedge} = \phi \hat{f}$ for each $f \in X$. Such a T commutes with translations and in many cases closed graph theorem implies that T is continuous. Consequently, such a T would define a multiplier from X to Y. This will be the case for pairs of spaces considered by us.

2.3.1 : The collection of all multipliers from X into Y will be denoted by $M(X,Y)$. The set of all functions ϕ on Γ which define elements $T \in M(X,Y)$ in the above manner will be denoted by $M_X^Y(\Gamma)$. We shall write $M(X,X) = M(X)$ and $M_X^X(\Gamma) = M_X(\Gamma)$. If G is infinite compact then $M(X,Y)$ and $M_X^Y(\Sigma)$ are defined similarly.

If X and Y are Segal algebras, then we have the following well known result, see for example [2].

THEOREM 2.3.2 : Let G be a locally compact abelian group and $S_i(G), i = 1,2$, be Segal algebras. Let $T: S_1(G) \to S_2(G)$ have this property : $T(f * g) = (Tf) * g = f * Tg$ for all $f,g \in S_1(G)$, then $T \in M(S_1, S_2)$.

We shall now give equivalent definitions of $T \in M(S_1, S_2)$. For proof of equivalence, see Burnham [2].

THEOREM : 2.3.3 : Let G be a locally compact abelian group, then the following are equivalent.

(i) $T \in M(S_1, S_2)$

(ii) There is a unique continuous function ϕ on Γ so that $(Tf)\hat{} = \phi \hat{f}$ for all $f \in S_1(G)$.

(iii) T is a bounded linear transformation from $S_1(G)$ into $S_2(G)$ commuting with convolutions.

2.3.4 : The norm of $\phi \in M_{S_1}^{S_2}(\Gamma)$ is defined as the corresponding operator norm of $T \in M(S_1, S_2)$. If $S_1(G) = S_2(G)$ or $S_1 = L_1(G)$, then $M_{S_1}^{S_2}(\Gamma)$ is a Banach algebra, with identity. If $\phi \in M_S(\Gamma)$, then norm of $\phi \in M_S(\Gamma)$ is denoted by $||\phi||_S$.

2.4 MULTIPLIERS OF TYPE (U,V) : Throughout the present section G will denote an infinite compact abelian group and Γ its dual group.

DEFINITION 2.4.1 : Let U and V be subsets of $C(\Gamma)$. An element ϕ in $C(\Gamma)$ is said to be an (U,V) multiplier if $\phi\psi \in V$ for all $\psi \in U$, i.e. if $\phi U \subseteq V$. An element ϕ is said to be an U-multiplier if it is an (U,U) multiplier.

For a subset A of $M(G)$, we will write $\hat{A}$ for the set $\{\hat{\mu} : \mu \in A\}$. When dealing with multipliers involving sets of

the form $\hat{A}$, we will drop the symbol $\hat{}$. For example, an $(\ell_p(\Gamma), L_p(G))$ multiplier is really an $(\ell_p(\Gamma), L_p(G)^{\wedge})$ multiplier and a $C(G)$-multiplier is a $C(G)^{\wedge}$-multiplier. We also frequently omit the symbols Γ and G and, for example, refer to (ℓ_p, L_p)- multiplier or C-multipliers.

The set of all (U,V)-multipliers will be denoted by the symbol $[U,V]$. The following results are well known; see [12, pp 367-389] .

THEOREM 2.4.2 : Let U and V be any of the spaces

(i) $\ell_p(\Gamma)$ $(1 \leq p \leq \infty)$, $c_o(\Gamma)$,

(ii) $L_p(G)$ $(1 \leq p \leq \infty)$, $C(G)$, $M(G)$.

Let ϕ be an (U,V)-multiplier. Define the mapping $T: U \to V$ by the following rules:

(iii) $T(\psi) = \phi\,\psi$ for $\psi \in U$ if U and V are chosen from (i)

(iv) $T(\psi)^{\wedge} = \phi\,\psi$ for $\psi \in U$ if U is chosen from (i) and V from (ii),

(v) $Tf = \phi\,\hat{f}$ for $f \in U$ [or $T(\mu) = \phi\,\hat{\mu}$ for $\mu \in M(G)$] if U is chosen from (ii) and V from (i)

(vi) $(Tf)^{\wedge} = \phi\,\hat{f}$ for $f \in U$ [or $(T(\mu))^{\wedge} = \phi\hat{\mu}$ for $\mu \in M(G)$]

if U and V are chosen from (ii). If U and V are given their usual norms, then T is a bounded linear transformation from U into V.

THEOREM 2.4.3 : Let $1 \leq q < p < \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Then

$$[\ell_p , \ell_q] = \ell_r$$

THEOREM 2.4.4 : For $1 < p \leq \infty$, we have

(i) $[M, L_p] = [L_1, L_p] = \hat{L}_p$,

for $1 \leq p < \infty$, we have

(ii) $[L_p, C] = [L_p, L_\infty] = \hat{L}_{p'}$

(iii) $[L_\infty, C] = \hat{L}_1$

THEOREM 2.4.5 : For $1 \leq p \leq \infty$, we have

(i) $[L_1, \ell_p] = \ell_p$, $[L_1, C_0] = \ell_\infty(\Gamma)$.

For $1 \leq p < \infty$, we have

(ii) $[\ell_p, L_\infty] = \ell_{p'}(\Gamma)$

The following theorems are proved by Edwards in [5]. We shall use these results frequently.

THEOREM 2.4.6 : For an element $\phi \in C(\Gamma)$, the following assertions are equivalent.

(i) $\phi \in \ell_2(\Gamma)$.

(ii) For every function $\varepsilon(\gamma) = \pm 1$ on Γ, $\varepsilon(\gamma)\,\phi(\gamma)$ is the Fourier transform of an integrable function on G.

(iii) For every function $\varepsilon(\gamma) = \pm 1$ on Γ, $\varepsilon(\gamma)\,\phi(\gamma)$ is the Fourier-Stieltjes transform of a measure on G.

THEOREM 2.4.7 : For an element $\phi \in C(\Gamma)$, the following are equivalent.

(i) $\phi \in \ell_1(\Gamma)$

(ii) For every function $\varepsilon(\gamma) = \pm 1$ on Γ, $\varepsilon(\gamma)\,\phi(\gamma)$ is the Fourier transform of a bounded function on G.

THEOREM 2.4.8 : Let G be an infinite compact abelian group and $1 \leq p < 2$. If a function ϕ on Γ is such that $\phi \hat{f} \in \ell_p(\Gamma)$ for every $f \in C(G)$, then $\phi \in \ell_{2p/2-p}(\Gamma)$.

REMARK 2.4.9 : Theorems 2.4.2 to 2.4.5 are in fact proved for infinite compact groups in [12,pp 367-389] . To get these theorems in this situation replace $\ell_p(\Gamma)$, $C_0(\Gamma)$ and $\ell_\infty(\Gamma)$ by $C_p(\Sigma)$, $C_0(\Sigma)$ and $C_\infty(\Sigma)$. $[C_0(\Sigma) = (E \in C(\Sigma) \mid \forall \varepsilon > 0, (\sigma \in \Sigma : \|E\|_{\phi_\infty} \geq \varepsilon)$ is finite$]$.

Theorems 2.4.6 and 2.4.8 also have valid analogues for infinite compact groups ([12, 36.13, 36.15]).

2.5 OPEN MAPPING THEOREM : The following theorem is a form of open mapping theorem which we shall use frequently. For proof see Kelley [15,p 99] .

THEOREM 2.5.1 : Let T be a continuous linear map of a complete pseudo-metrizable linear topological space E into a Hausdorff linear topological space F. If the range of T is of second category in F, then T maps E onto F, F is complete and metrizable and T is an open mapping.

CHAPTER - III

PROPER INCLUSIONS IN A_p ALGEBRAS

3.1 INTRODUCTION : In [16] and [17, p 197] Larsen stated without proof that algebras $A_p(G)$ $(1 \leq p < \infty)$ are distinct for distinct p unless G is discrete, in which case $A_p(G) = L_1(G) \; \forall \; p$. In a private communication Larsen told us that his proof of this assertion was fallacious and he gave a proof of the fact that $A_p(G) \subsetneqq A_q(G)$, $p < q$, in case $G = R$ or T, or G is infinite compact and $1 \leq p < 2$. In this chapter we show that $\bigcup_{1 \leq p < q} A_p(G) \subsetneqq A_q(G)$ provided G is a nondiscrete locally compact abelian group or an infinite compact group. Furthermore, it is shown that $\bigcup_{1 \leq p < q} A_p(G)$ is a $\| \; \|_{A_q}$ dense linear subspace of $A_q(G)$ of category I and the ideals generated by $A_p(G) * A_q(G)$ and $S(G) * L_p(G)$ are properly contained in $A_p(G)$ and $L_p(G)$ respectively, provided in the latter situation G is infinite compact abelian and $(S(G))^{\wedge} \subseteq L_{p_0}(\Gamma)$ for some $0 < p_0 < \infty$. These assertions answer the questions raised by Larsen in 'B' and 'K' of [18]. Part of the contents of this chapter has appeared in [30].

Theorem 3.1.16 is the main result of this chapter which precisely states the following:

THEOREM 3.1.16 : Let G be a nondiscrete locally compact abelian or an infinite compact group and $1 \leq p < q < \infty$. Then

$\bigcup\limits_{1 \leq p < q} A_p(G)$ is a $\|\ \|_{A_q}$ dense linear subspace of $A_q(G)$ of category I.

We accomplish Theorem 3.1.16 in three steps.

STEP 1 : We prove $A_p(G) \subsetneq A_q(G)$ provided $1 \leq p < q < \infty$ and $G = T$, $G = R$ or $G = F$ an infinite compact group.

STEP 2 : Using step 1 and structure of a locally compact abelian group we derive the result for any nondiscrete locally compact abelian group.

STEP 3 : Using steps 1 and 2 and open mapping theorem we conclude the result.

The complete proof of Theorem 3.1.16 contains several intermediary results which are of interest for their own sake. Now onwards, throughout this chapter we have $1 \leq p < q < \infty$ unless otherwise stated.

PROPOSITION 3.1.1 : Let G be a nondiscrete locally compact abelian group and $1 \leq p < 2$, then $A_p(G) \subsetneq A_q(G)$.

PROOF : Since G is nondiscrete, Γ is non-compact. Let U be a symmetric neighbourhood of 0 in Γ such that $\bar{U}$ is compact. Choose a sequence $\{\gamma_n\}$ in Γ such that $\gamma_i + U + U$ is disjoint from $\gamma_j + U + U$ if $i \neq j$. Let

$$g = \chi_U \quad \text{and} \quad h = \sum_{k=1}^{\infty} \frac{1}{k^{1/p}} \chi_{\gamma_k + U + U}$$

Then g and $h \in L_2(\Gamma)$. Hence there exists $f \in L_1(G)$ such

that $\hat{f} = g * h$. Moreover, $g \in L_1(\Gamma)$ and

$$||h||_q^q = \int_\Gamma |h|^q \, d\eta \leq \eta(U+U) \sum_{k=1}^{\infty} \frac{1}{k^{q/p}} < \infty$$

Therefore $h \in L_q(\Gamma)$ and $f \in A_q(G)$; but $f \notin A_p(G)$ as can be seen from the following:

Let $\tau \in \gamma_k + U$, then

$$g * h(\tau) = \int_\Gamma \chi_U(\tau - y) \sum_{k=1}^{\infty} \frac{1}{k^{1/p}} \chi_{\gamma_k+U+U}(y) \, d\eta(y)$$

$$= 0 \text{ unless } \tau - y \in U$$

Now $\tau - y \in U$ implies $y - \tau \in -U = U$ or $y \in \gamma_k+U+U$. Hence

$$(3.1.1a) \qquad g * h(\tau) = \frac{1}{k^{1/p}} \eta(U)$$

Since $\gamma_i + U$ is disjoint from $\gamma_j + U$ if $i \neq j$, it follows from (3.1.1a) that $f \notin A_p(G)$.

COROLLARY 3.1.2 : Let $G = T$, the circle group. Then $A_p(G) \subsetneq A_q(G)$.

PROOF : If $1 \leq p < 2$, then the result follows from Proposition 3.1.1. If $p \geq 2$, then $q > 2$ and q' lies between 1 and 2. It is known [7, p 147] that there exists $f \in L_{q'}(G)$ such that $\hat{f} \notin L_p(\Gamma)$. Such a function $f \in A_q(G)$ but $f \notin A_p(G)$.

PROPOSITION 3.1.3 : Let $G = R$, the real line. Then $A_p(G) \subsetneq A_q(G)$.

PROOF : Since $p < q$, it follows that $A_p(G) \subseteq A_q(G)$. Moreover,

$$||f||_{A_q} = ||f||_1 + ||\hat{f}||_q \leq 2 \, ||f||_{A_p} \; \forall f \in A_p(G)$$

Therefore the assumption that $A_p(G) = A_q(G)$ would lead to the existence of a constant $K > 0$ such that

$$(3.1.3a) \qquad ||\hat{f}||_p \leq K[||f||_1 + ||\hat{f}||_q] \quad \forall\, f \in A_p(G)$$

We shall show that (3.1.3a) leads to a contradiction. For this purpose consider the function

$$\Delta_\alpha(x) = \begin{cases} 1 - |x|/\alpha, & |x| \leq \alpha \\ 0 & |x| > \alpha \end{cases}$$

and let $f_\alpha = \check{\Delta}_\alpha$, then

$$f_\alpha(x) = \int_{-\infty}^{\infty} \Delta_\alpha(t)\, e^{itx}\, dt = 2 \int_0^\alpha (1 - \frac{t}{\alpha}) \cos t x \, dt$$

$$= \frac{2(1-\cos \alpha x)}{\alpha x^2} = 2\pi\delta_\alpha(x)$$

where $\delta_\alpha(x) = \dfrac{1}{\pi} \dfrac{(1-\cos \alpha x)}{\alpha x^2}$

Since δ_α and $\Delta_\alpha \in L_1$ and Δ_α is continuous on $(-\infty, \infty)$, therefore $\Delta_\alpha(x) = \hat{\delta}_\alpha(x)$. Now $||f_\alpha||_1 = 2\pi\Delta_\alpha(0) = 2\pi$. Moreover, $\hat{f}_\alpha = 2\pi\Delta_\alpha$. Therefore

$$||\hat{f}_\alpha||_p = 2\pi \left(\int_{-\infty}^{\infty} |\Delta_\alpha(t)|^p dt \right)^{1/p} = 2\pi\, 2^{1/p} \left(\int_0^\alpha (1 - \frac{t}{\alpha})^p dt \right)^{1/p}$$

$$(3.1.3b) \qquad = 2\pi \left[\frac{2}{p+1}\right]^{1/p} \alpha^{1/p}$$

From (3.1.3a) and (3.1.3b) it follows that

$$(3.1.3c) \qquad 2\pi \left(\frac{2}{p+1}\right)^{1/p} \alpha^{1/p} \leq 2K\pi + 2K\pi \left(\frac{2}{q+1}\right)^{1/q} \alpha^{1/q}$$

Dividing by $\alpha^{1/q}$ on both sides of (3.1.3c) we get

$$(3.1.3d) \qquad 2\pi\left(\frac{2}{p+1}\right)^{1/p} \alpha^{1/p - 1/q} \leq 2K\pi\,\alpha^{-1/q} + 2K\pi\left(\frac{2}{q+1}\right)^{1/q}$$

Now as α tends to infinity in (3.1.3d) we see that the right hand side remains bounded while the left hand side tends to infinity. This contradiction establishes the proposition.

PROPOSITION 3.1.4 : Let G be an infinite compact totally disconnected abelian group. Then $A_p(G) \subsetneqq A_q(G)$.

PROOF : As in the proof of the Proposition 3.1.3, the assumption that $A_p(G) = A_q(G)$ would lead to the existence of a constant $K > 0$ such that

$$(3.1.4a) \qquad ||\hat{f}||_p \leq K\,||f||_1 + K\,||\hat{f}||_q \;\forall\, f \in A_p(G).$$

We shall show that (3.1.4a) leads to a contradiction. Since G is compact and totally disconnected, there exists a neighbourhood basis $\{V_\alpha\}_{\alpha\in I}$ of 0 consisting of open and closed subgroups of G [11, p 62] . Since G is infinite compact it follows that $\lim\limits_{\alpha} \lambda(V_\alpha) = 0$. Let $\lambda_\alpha = \lambda(V_\alpha)$ and let X_α denote the annihilator subgroup of V_α. Since V_α is open and closed, it follows that X_α is finite. Let n_α be the number of points in X_α and let $f_\alpha = \chi_{V_\alpha}$. Then

$$||f_\alpha||_1 = \lambda_\alpha\;, \quad \hat{f}_\alpha(\gamma) = \lambda_\alpha\,\chi_{X_\alpha}(\gamma)$$

Also $||f_\alpha||_2^2 = \lambda_\alpha$. Therefore by the Plancherel theorem we get

$$\lambda_\alpha = ||f_\alpha||_2^2 = \sum_{\gamma \in \Gamma} |\hat{f}_\alpha(\gamma)|^2 = \lambda_\alpha^2 \; n_\alpha$$

Hence $n_\alpha = 1/\lambda_\alpha$. Now, for $1 \leq p < \infty$, we have

$$||\hat{f}_\alpha||_p = (\sum_{\gamma \in \Gamma} |\lambda_\alpha \chi_{X_\alpha}|^p)^{1/p} = \lambda_\alpha n_\alpha^{1/p} = \lambda_\alpha^{1/p'}$$

Now from (3.1.4a) it follows that

(3.1.4 b) $$\lambda_\alpha^{1/p'} \leq K \lambda_\alpha + K \lambda_\alpha^{1/q'}$$

Dividing both sides of (3.1.4b) by $\alpha^{1/q'}$ we get

(3.1.4c) $$\lambda_\alpha^{1/p' - 1/q'} \leq K \lambda_\alpha^{1/q} + K$$

Taking the limit in (3.1.4c) we see that the right hand side remains bounded while the left hand side tends to infinity, because $\frac{1}{p'} < \frac{1}{q'}$ and $\lambda_\alpha \to 0$. This contradiction yields the result.

PROPOSITION 3.1.5 : Let G_1 and G_2 be locally compact abelian groups and $G = G_1 \times G_2$. If $A_p(G_i) \neq A_q(G_i)$ for some $i=1,2$, then $A_p(G) \neq A_q(G)$.

PROOF : We may assume that $A_p(G_1) \neq A_q(G_1)$ (the proof in the other case is exactly similar). Choose $f \in A_q(G_1)$ such that $f \notin A_p(G_1)$. Let g be any non zero function belonging to $A_p(G_2)$. Define $h(x,y) = f(x)\, g(y)$ for $(x,y) \in G_1 \times G_2$. Then $\hat{h}(\gamma,\eta) = \hat{f}(\gamma)\, \hat{g}(\eta)$. Now we claim : $\hat{h} \in L_p(\Gamma)$ if and only if $\hat{f} \in L_p(\Gamma_1)$ and $\hat{g} \in L_p(\Gamma_2)$. Our claim is trivial once we observe that

$$\int_{\Gamma} |\hat{h}(\gamma,\eta)|^p d(\gamma \times \eta) = \int_{\Gamma} |\hat{f}(\gamma)\ \hat{g}(\eta)|^p d(\gamma \times \eta)$$

$$= \int_{\Gamma} |\hat{f}(\gamma)|^p |\hat{g}(\eta)|^p d(\gamma \times \eta)$$

$$= \int_{\Gamma_1} |\hat{f}(\gamma)|^p\, d\gamma \int_{\Gamma_2} |\hat{g}(\eta)|^p d\eta\ .$$

Last step follows because of Fubini's theorem. Hence the function $f(x)\ g(y) = h(x,y) \in A_q(G)$ but $h(x,y) \notin A_p(G)$.

Let H be a closed subgroup of a locally compact abelian group G. It is known [27, p 131] that if $S(G)$ is a Segal algebra on G, then $\Pi_H(S(G))$ becomes a Segal algebra on G/H. The following proposition is interesting because it shows that $\Pi_H(A_p(G)) = A_p(G/H)$ under the hypothesis that H is compact.

PROPOSITION 3.1.6 : Let H be a compact subgroup of a locally compact abelian group G. Then $\Pi_H(A_p(G)) = A_p(G/H)$.

PROOF : Let Λ be the annihilator of H. Since H is compact, Λ is open. It can be easily seen that for $f \in L_1(G)$, $(\Pi_H(f))^{\wedge}(\gamma) = \hat{f}(\gamma)|_{\Lambda}$. Therefore $f \in A_p(G)$ implies $\Pi_H(f) \in A_p(G/H)$ and so $\Pi_H(A_p(G)) \subseteq A_p(G/H)$. To prove the other inclusion, take $f' \in A_p(G/H)$; then there exists $f \in L_1(G)$ such that $\Pi_H(f) = f'$. Now consider the function $g = m_H * f$ where m_H is the normalised Haar measure on H and is considered as a bounded measure on the whole group G by taking it to be zero outside H. Since $\hat{m_H} = \chi_{\Lambda}$, it follows that $\Pi_H(g) = f'$. Indeed ,

$$(\Pi_H(g))^{\wedge} = \hat{g}|_{\Lambda} = \hat{m_H} \hat{f}|_{\Lambda} = \hat{f}|_{\Lambda} = (\Pi_H(f))^{\wedge} = \hat{f}' ;$$

hence $\Pi_H(g) = f'$. Further, since Λ is open and $\hat{g} = \hat{f}'$ on Λ and $\hat{g} = 0$ outside Λ, it follows that $g \in A_p(G)$.

REMARK 3.1.7 : Larsen in 'G' of [18] asked if Proposition 3.1.6 is true for closed subgroups of G. Krogstad in [13] has shown (by taking $G = R$ and $H = 2\Pi Z$) that the assertion of Proposition 3.1.6 is not valid for all closed subgroups of G. Thus the condition of compactness on the subgroup H of G is essential and can not be dropped in the above proposition.

THEOREM 3.1.8 : Let G be an infinite compact abelian group. Then $A_p(G) \subsetneq A_q(G)$.

PROOF : We have already proved the theorem for totally disconnected G in Proposition 3.1.4. So suppose G is not totally disconnected. Then the dual group Γ has an element of infinite order [28, p 47]. Therefore it contains Z (the group of integers) as a subgroup. Let H be the annihilator of this subgroup. Then the dual group of G/H is isomorphic to Z and G/H is isomorphic to T. By Corollary 3.1.2 it follows that $A_p(G/H) \subsetneq A_q(G/H)$. The theorem now follows by Proposition 3.1.6.

REMARK 3.1.9 : It is proper here to point out that we had obtained the proof of Theorem 3.1.8 before we came to know that for $p \geq 2$ it can be derived from Theorem 2.b of Fournier [8]. Indeed, Theorem 2.b of Fournier tells that if G is infinite compact

abelian group and $1 \leq p \leq 2$, then

$$(L_p(G))^{\wedge} \not\subseteq \bigcup_{q < p'} L_q(\Gamma)$$

This shows that for $p \geq 2$ there exists a function $f \in L_{q'}(G)$ such that $\hat{f} \notin \ell_p(\Gamma)$. Now it is clear that $f \in A_q(G)$ but $f \notin A_p(G)$. Fournier also remarks in [8] that Theorem 2.b is true even if the group G is infinite compact, implying thereby that for $p \geq 2$, $A_p(G) \subsetneqq A_q(G)$. But for $1 \leq p < 2$ and G an infinite compact group we can easily see that $A_p(G) \subsetneqq A_q(G)$. Indeed, let $\{\sigma_n\}$ be an infinite sequence in Σ. Define $E = (E_\sigma) \in C(\Sigma)$ by

$$E_\sigma = \begin{cases} \dfrac{1}{n^{1/p}} I_{d\sigma_n} & \text{if } \sigma = \sigma_n \\ 0 & \text{if } \sigma \neq \sigma_n \end{cases}$$

It is clear that $E \in C_2(\Sigma)$ and $C_q(\Sigma)$ but $E \notin C_p(\Sigma)$. But $E \in C_2(\Sigma)$ implies there exists a $f \in L_2(G) \subseteq L_1(G)$ such that $\hat{f} = E$. Then it is obvious that f is the required function. Now the following theorem is clear because of the above remark.

THEOREM 3.1.10 : Let G be an infinite compact group. Then $A_p(G) \subsetneqq A_q(G)$.

PROPOSITION 3.1.11 : Let H be an open subgroup of a locally compact abelian group G. Let $f \in L_1(H)$. Define g on G as follows.

$$g(x) = \begin{cases} f(x) & \text{if } x \in H \\ 0 & \text{if } x \notin H \end{cases}$$

Then $g \in L_1(G)$ and $\hat{g} \in L_p(\Gamma)$ if and only if $\hat{f} \in L_p(\Gamma/\Lambda)$, where Λ is the annihilator of H.

PROOF : It is obvious that $g \in L_1(G)$. Let

$$F = \{\phi \in L_1(\Gamma) \mid \phi \text{ is almost every where constant on each coset of } \Lambda\}$$

and let $g' : \Gamma \to \Gamma/\Lambda$ be the quotient map. Then it follows from [12, p 95(28.55)] that $h \to h \circ g'$ is a Banach algebra isomorphism of $L_1(\Gamma/\Lambda)$ onto F. Since g is zero outside H, $\hat{g}$ is constant on cosets of Λ. Moreover,

$$\hat{f} \circ g'(\gamma) = \int_H f(x)\ (-x, g'(\gamma))dx$$

$$= \int_G g(x)\ (-x, g'(\gamma))dx = \hat{g}\ (g'(\gamma)) = \hat{g}(\gamma)$$

Last step follows because $\hat{g}$ is constant on cosets of Λ. Also ,

$$|\hat{f}|^p \circ g'(\gamma) = |\hat{f}(g'(\gamma)|^p = |\hat{g}(\gamma)|^p = |\hat{g}|^p(\gamma) .$$

Hence it follows that $\hat{g} \in L_p(\Gamma)$ if and only if $\hat{f} \in L_p(\Gamma/\Lambda)$.

COROLLARY 3.1.12: Let H be an open subgroup of a locally compact abelian group G such that $A_p(H) \subsetneq A_q(H)$, then $A_p(G) \subsetneq A_q(G)$.

PROOF : Let $f \in A_q(H)$ such that $f \notin A_p(H)$. Define g as in Proposition 3.1.11. Then Proposition 3.1.11 implies that $g \in A_q(G)$ but $g \notin A_p(G)$.

THEOREM 3.1.13 : Let G be a nondiscrete locally compact abelian group. Then $A_p(G) \subsetneqq A_q(G)$.

PROOF : [28, 2.4.1] implies that there exists an open subgroup H of G such that $H = R^n \times F$, where n is a non-negative integer and F a compact abelian group. If $n > 0$, then $A_p(H) \subsetneqq A_q(H)$ by Propositions 3.1.3 and 3.1.5. If $n = 0$, then since G is nondiscrete and H is open, it follows that F is an infinite compact abelian group. Therefore $A_p(H) \subsetneqq A_q(H)$. The theorem now follows by Corollary 3.1.12.

Burnham, Krogstad and Larsen in [4, cor 2] have shown that, for $p > 2$, $A_p(G)$ is not a prime ideal in $M(G)$ provided G is a nondiscrete locally compact abelian group. Their proof uses the facts that $A_p(G) \subsetneqq L_1(G)$ and for $p > 2$ there exists a singular measure on G whose Fourier transform belongs to $L_p(\Gamma)$. However, by using Theorem 3.1.13 we shall show that $A_p(G)$ is not a prime ideal in $L_1(G)$ for all $1 \leq p < \infty$.

COROLLARY 3.1.14 : Let G be a nondiscrete locally compact abelian or an infinite compact group. Then for each $p, 1 \leq p < \infty$, $A_p(G)$ is not a prime ideal in $L_1(G)$.

PROOF : It is evident that $A_{2p}(G) * A_{2p}(G) \subseteq A_p(G)$. Also $A_p(G) \neq A_{2p}(G)$ by Theorem 3.1.13 and 3.1.10; therefore corollary follows.

The following corollary answers the question raised by Larsen in 'B' of [18].

COROLLARY 3.1.15 : Let G be a nondiscrete locally compact abelian or an infinite compact group. Then, $A_p(G)$ is a $\| \|_{A_q}$ dense linear subspace of $A_q(G)$ of category I.

PROOF: Since integrable functions whose Fourier transforms have compact support are dense in $A_q(G)$ for every q, $1 \leq q < \infty$, therefore $A_p(G)$ is a $\| \|_{A_q}$ dense linear subspace of $A_q(G)$. To prove that $A_p(G)$ is of category I in $A_q(G)$, let $i: A_p(G) \to A_q(G)$ be the identity map; clearly i is continuous. Since $A_p(G) \subsetneq A_q(G)$ by Theorems 3.1.13 and 3.1.10, it follows from Theorem 2.5.1 that $A_p(G)$ is of category I in $A_q(G)$.

THEOREM 3.1.16 : Let G be a nondiscrete locally compact abelian or an infinite compact group. Then $\bigcup_{1 \leq p < q} A_p(G)$ is a $\| \|_{A_q}$ dense subspace of $A_q(G)$ of category I.

PROOF : Clearly $\bigcup_{1 \leq p < q} A_p(G)$ is a $\| \|_{A_q}$ dense subspace of $A_q(G)$. Choose a sequence $\{p_n\}$ increasing to q. It can be easily seen that $\bigcup_{n=1}^{\infty} A_{p_n}(G) = \bigcup_{1 \leq p < q} A_p(G)$. For each n, $A_{p_n}(G)$ is of first category in $A_q(G)$ (Corollary 3.1.15). Therefore, $\bigcup_{1 \leq p < q} A_p(G)$, being a countable union of sets of first category, is a set of first category.

Let G be a nondiscrete locally compact abelian group and $S(G)$ be a Segal algebra on G. Larsen in 'K' of [18] asked if the ideals generated by $A_p(G) * A_q(G)$ and $S(G) * L_p(G)$ equal $A_p(G)$ and

$L_p(G)$ respectively. The first question of Larsen we shall solve fully by showing that the ideal generated by $A_p(G) * A_q(G)$ is properly contained in $A_p(G)$. This fact is also proved by Yap [35]. But we do not know the way Yap has proved this result. We shall prove this result following an idea due to Burnham [1] and Wang [31]. In fact we shall prove a more general result. The latter question of Larsen we have solved under some conditions. The conditions are

(i) Group G is infinite compact abelian

(ii) $S(G)^{\wedge} \subseteq L_{p_0}(\Gamma)$ for some p_0, $0 < p_0 < \infty$.

THEOREM 3.1.17 : Let $S_1(G)$ and $S_2(G)$ be two Segal algebras on G such that $(S_1(G))^{\wedge} \subseteq L_{p_0}(\Gamma)$, $(S_2(G))^{\wedge} \not\subseteq L_{r_0}(\Gamma)$ for some p_0 and r_0, $0 < p_0, r_0 < \infty$. Then the ideal generated by $S_1(G) * S_2(G)$ is properly contained in $S_2(G)$.

PROOF : It is clear that $I(S_1(G) * S_2(G))$ (the ideal generated by $S_1(G) * S_2(G)) \subseteq S_2(G)$. If possible, suppose $I(S_1(G)*S_2(G)) = S_2(G)$. Let $f \in S_2(G)$. Then,

$$f = \sum_{i=1}^{n} \alpha_i (f_i * f_i'), \alpha_i \in ¢, f_i \in S_1(G), f_i' \in S_2(G). \tag{3.1.17a}$$

Since each $f_i' \in S_2(G)$, therefore

$$f_i' = \sum_{j=1}^{m_i} \beta_{ij} (f_{ij} * f_{ij}'), i = 1,2,\ldots n \tag{3.1.17b}$$

where $\beta_{ij} \in ¢$, $f_{ij} \in S_1(G)$ and $f_{ij}' \in S_2(G)$.

Substituting (3.1.17b) in (3.1.17a) we get

$$(3.1.17c) \qquad f = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \alpha_i \, \beta_{ij} \, f'_{ij} * (f_i * f_{ij}) .$$

Since $\hat{f}_i^{\,p_o/2}$, $\hat{f}_{ij}^{\,p_o/2} \in L_2(\Gamma)$, therefore by Hölder's inequality and (3.1.17c) we get $\hat{f} \in L_{p_o/2}(\Gamma)$. Since f is an arbitrary element in $S_2(G)$, therefore we get $S_2(G)^{\wedge} \subseteq L_{p_o/2}(\Gamma)$. Continuing this process n times we get $S_2(G)^{\wedge} \subseteq L_{p_o/2^n}(\Gamma)$. Choose n so large that $p_o/2^n < r_o$. Therefore $S_2(G)^{\wedge} \subseteq L_{r_o}(\Gamma)$, a contradiction.

COROLLARY 3.1.18 : $I(A_p(G) * A_q(G)) \subsetneqq A_p(G)$.

PROOF : If $p > 1$, then the corollary follows immediately from Theorem 3.1.17, since $(A_p(G))^{\wedge} \nsubseteq L_1(\Gamma)$ (Theorem 3.1.13). If $p = 1$, then by choosing $h = \sum_{k=1}^{\infty} \frac{1}{k^2} \chi_{\gamma_k + U + U}$ ($\{\gamma_k\}$ and U are as in Proposition 3.1.1) we can show that $A_{1/2}(G) \subsetneqq A_1(G)$. This fact and Theorem 3.1.17 now completes the proof. $[A_{1/2}(G) = (f \in L_1(G) \mid \hat{f} \in L_{1/2}(\Gamma))]$.

The following corollary is immediate from Theorem 3.1.17.

COROLLARY 3.1.19 : Let $1 \leq p < \infty$, then $I(S(G) * L_p(G)) \subsetneqq L_p(G)$ under the conditions, mentioned earlier.

REMARK 3.1.20 : It can be easily seen that Theorem 3.1.17 and Corollaries 3.1.18 and 3.1.19 are valid for infinite compact groups.

CHAPTER - IV

ON MULTIPLIERS FROM $A_p(G)$ TO $A_q(G)$ ALGEBRAS

INTRODUCTION : It is known [17, pp 204-207] that $M_{A_p}(\Gamma) = M(G)^{\wedge}$, $1 \leq p < \infty$, in case G is a noncompact locally compact abelian group and if G is an infinite compact abelian group, $1 \leq p \leq 2$, then $M_{A_p}(\Gamma) = C(\Gamma)$. These results show that $M_{A_p}(\Gamma)$ does not characterize $A_p(G)$-algebras in the sense that $A_p(G) \neq A_q(G)$ need not imply $M_{A_p}(\Gamma) \neq M_{A_q}(\Gamma)$. We ask the following natural question.

QUESTION : Does $M_{A_p}(\Gamma)$ characterize $A_p(G)$-algebra if G is an infinite compact abelian group and $2 < p < \infty$?

Affirmative answer to this question will be given in Section 1 of this chapter.

In Section 2 we study multipliers between $L_1(G)$, $A_p(G)$ and $A_q(G)$ algebras. Considerable amount is known about multipliers from $A_p(G)$ to $A_p(G)$ and $L_1(G)$ to $A_p(G)$ algebras. See for example [17] , [14] and [3] . Combining this with the known fact that $M_{A_p}^{L_1}(\Gamma) = M_{A_p}^{A_q}(\Gamma) = M_{A_p}(\Gamma)$ provided $1 \leq p \leq q < \infty$, it follows that interesting case is when $1 \leq q < p < \infty$. We show that when G is a nondiscrete locally compact abelian group and $1 \leq q < p < \infty$, then

$$(B_{pq/p-q}(G))^{\wedge} \subsetneqq M_{A_p}^{A_q}(\Gamma) \subsetneqq (M(G))^{\wedge}, \text{ if } G \text{ is non-compact.}$$

However, if G is an infinite compact abelian group and $1 \leq q < p \leq 2$, then $M_{A_p}^{A_q}(\Gamma) = \ell_{pq/p-q}(\Gamma)$. Moreover, $\ell_{pq/p-q}(\Gamma)$ is the largest $\ell_r(\Gamma)$-space contained in $M_{A_p}^{A_q}(\Gamma)$ and $M_{A_p}^{A_q}(\Gamma) \subsetneqq \ell_{2q/2-q}(\Gamma)$ for $1 \leq q \leq 2 < p$. Furthermore, $\ell_{pq/p-q}(\Gamma)$ need not be contained in $M_{A_p}^{A_q}(\Gamma)$ for certain values of p and q satisfying $2 < q < p < \infty$. However $(B_{pq/p-q}(G))^{\wedge} \subsetneqq M_{A_p}^{A_q}(\Gamma)$ for all $2 < q < p < \infty$.

Section 3 deals with restriction of $M_{A_p}^{A_q}(\Gamma)$ to closed subgroups Γ_0 of Γ and the corresponding extension problem. We prove that if G is noncompact, then $M_{A_p}(\Gamma)|\Gamma_0 = M_{A_p}(\Gamma_0)$ unless Γ_0 is a closed discrete subgroup of Γ, in which case $M_{A_p}(\Gamma)|\Gamma_0 \subsetneqq M_{A_p}(\Gamma_0)$. But if G is an infinite compact abelian group, then we have $M_{A_p}^{A_q}(\Gamma)|\Gamma_0 = M_{A_p}^{A_q}(\Gamma_0)$, $1 \leq p,q < \infty$.

4.1 PROPER INCLUSIONS IN A_p-MULTIPLIERS : By proving the existence of sets of uniqueness for $L_p(G)$, $1 \leq p < 2$, Figa-Talamanca and Gaudry have proved in [9] that $M(L_p(G)) \subsetneqq M(L_2(G))$ when $1 \leq p < 2$ and G is any infinite locally compact

abelian group, thus improving an earlier result of R.E. Edwards which required G to satisfy the extra condition of possessing an infinite discrete subgroup. The authors of [9] then employ the Riesz-Thorin convexity theorem to prove that $M_{L_p}(\Gamma) \cap C_o(\Gamma) \subsetneqq M_{L_q}(\Gamma) \cap C_o(\Gamma)$ when $1 \leq p < q \leq 2$, unless G is finite.

J.F. Price in [26] generalizes the basic result $M(L_p(G)) \subsetneqq M(L_q(G))$ when $1 \leq p < q \leq 2$. Price shows, using a partly constructive proof, that when G is infinite,

$$\bigcup_{1 \leq q < p} M(L_q(G)) \subsetneqq M(L_p(G)) \subsetneqq \bigcap_{p < q \leq 2} M(L_q(G))$$

for $1 < p < 2$, with the first inclusion remaining strict when $p = 2$ and the second inclusion remaining strict when $p = 1$ (similar results also hold for $2 \leq p \leq \infty$).

In view of these results one naturally asks for the corresponding results for A_p-multipliers. The following theorem answers some corresponding questions for A_p-multipliers and it turns out that proof of this theorem is much simpler than the corresponding theorem for $L_p(G)$.

THEOREM : 4.1.1 : Let G be an infinite compact abelian group and $1 \leq q < \infty$, $2 < p < \infty$, $p > q$, then

(i) $M_{A_p}(\Gamma) \cap C_o(\Gamma) \subsetneqq M_{A_q}(\Gamma) \cap C_o(\Gamma)$

(ii) $\bigcup_{p > q} M_{A_p}(\Gamma) \subsetneqq M_{A_q}(\Gamma)$.

PROOF : (i) If $1 \leq q \leq 2$, then $M_{A_q}(\Gamma) = C(\Gamma)$. But there exists a function $\phi \in C_0(\Gamma)$ such that $\phi \notin M_{A_p}(\Gamma)$, see [17, p 208].

Let us now suppose that $2 < q < p < \infty$. Let $r = \frac{2q}{q-2}$; then $r > 2$ and $q = \frac{2r}{r-2}$. By Theorem 3.1.8, there exists a function $f \in A_p(G)$ such that $\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^{2r/r-2} = \infty$. Then by Theorem 2.4.3 there exists a $\psi \in \ell_r(\Gamma)$ such that $\psi \hat{f} \notin \ell_2(\Gamma)$. Hence by Theorem 2.4.6 there exists a function $\varepsilon(\gamma) = \pm 1$ on Γ such that $\varepsilon(\gamma)\psi(\gamma)\hat{f}(\gamma)$ is not the Fourier transform of any integrable function. Let $\phi(\gamma) = \varepsilon(\gamma)\psi(\gamma)$. Then it is clear that $\phi \notin M_{A_p}(\Gamma)$ but $\phi \in \ell_r(\Gamma)$. Therefore,

(4.1.1a) $\qquad \ell_r(\Gamma) \not\subseteq M_{A_p}(\Gamma)$.

Now we shall show that $\ell_r(\Gamma) \subseteq M_{A_q}(\Gamma)$. Let $\phi \in \ell_r(\Gamma) = \ell_{2q/q-2}(\Gamma)$ and $f \in A_q(G)$. Since $\frac{q}{q-2} = (\frac{q}{2})'$, therefore by Hölder's inequality we get

$$\sum_{\gamma \in \Gamma} |\phi \hat{f}(\gamma)|^2 \leq \underset{< \infty}{\left(\sum_{\gamma \in \Gamma} |\phi(\gamma)|^{2q/q-2}\right)^{q-2/q} \left(\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^q\right)^{2/q}}$$

Consequently, $\phi \hat{f} \in \ell_2(\Gamma) \subseteq \ell_q(\Gamma)$. Hence there exists $g \in L_1(G)$ such that $\hat{g} = \phi \hat{f}$ and thus $\phi \in M_{A_q}(\Gamma)$. Therefore,

(4.1.1b) $\qquad \ell_r(\Gamma) \subseteq M_{A_q}(\Gamma)$.

(4.1.1a) and (4.1.1b) imply that

(4.1.1c) $$M_{A_p}(\Gamma) \cap c_o(\Gamma) \subsetneqq M_{A_q}(\Gamma) \cap c_o(\Gamma) .$$

(ii) Now to prove $\bigcup_{p>q} M_{A_p}(\Gamma) \neq M_{A_q}(\Gamma)$, we recall that if $1 \le r < \infty$, then $M_{A_r}(\Gamma)$ is a Banach space with norm of $\phi \varepsilon M_{A_r}(\Gamma)$ defined as the operator norm of T, where T is the operator associated with ϕ such that $(Tf)^{\wedge} = \phi \hat{f} \; \forall f \varepsilon A_r(G)$.
Thus $M_{A_p}(\Gamma)$ and $M_{A_q}(\Gamma)$ are Banach spaces. Moreover, there exists a constant $K > 0$ such that

(4.1.1d) $$||\phi||_{A_q} \le K \, ||\phi||_{A_p} .$$

Let $i : M_{A_p}(\Gamma) \to M_{A_q}(\Gamma)$ be the identity map. Then (4.1.1d) tells us that i is a continuous map. Therefore it follows from Theorem 2.5.1 that $M_{A_p}(\Gamma)$ is of first category in $M_{A_q}(\Gamma)$. Let $\{p_n\}$ be a decreasing sequence such that $p_n \to q$. Then it can be easily seen that

$$\bigcup_{p>q} M_{A_p}(\Gamma) = \bigcup_{n=1}^{\infty} M_{A_{p_n}}(\Gamma) .$$

This shows that $\bigcup_{p>q} M_{A_p}(\Gamma)$ is of first category in $M_{A_q}(\Gamma)$.

Now it is clear that

(4.1.1e) $$\bigcup_{p>q} M_{A_p}(\Gamma) \subsetneqq M_{A_q}(\Gamma) .$$

For if equality holds in (4.1.1e), then it will follow that $M_{A_q}(\Gamma)$, a Banach space, is of first category in itself, a contradiction by Baire's category theorem.

REMARK 4.1.2 : The assertion of Theorem 4.1.1 holds even if G is an infinite compact group. The proof is exactly the same as for the abelian case. The results needed in the arguments in Theorem 4.1.1 are given in Theorem 3.1.10 and Remark 2.4.9.

4.2 MULTIPLIERS FROM $A_p(G)$ INTO $A_q(G)$ ALGEBRAS : In this section we shall study $M_{A_p}^{A_q}(\Gamma)$, $1 \leq p, q < \infty$. Let us first observe a simple but important fact about $M_{A_p}^{A_q}(\Gamma)$. For a fixed q, $1 \leq q < \infty$, $M_{A_p}^{A_q}(\Gamma)$ is non-increasing as the index p is increasing; that is, $p_1 < p_2$ implies $M_{A_{p_2}}^{A_q}(\Gamma) \subseteq M_{A_{p_1}}^{A_q}(\Gamma)$. But for fixed p, $1 \leq p < \infty$, $M_{A_p}^{A_q}(\Gamma)$ is non-decreasing as the index q is increasing; that is, $q_1 < q_2$ implies $M_{A_p}^{A_{q_1}}(\Gamma) \subseteq M_{A_p}^{A_{q_2}}(\Gamma)$. We shall use these facts frequently.

The following results, till Remark 4.2.6, are either straight forward or known. We state these results for the sake of completeness.

Arguing as in Larsen [17, p 201] we get the following proposition.

PROPOSITION 4.2.1 : Let G be a locally compact abelian group and $1 \leq p \leq q < \infty$, then

$$M_{A_p}^{L_1}(\Gamma) = M_{A_p}^{A_q}(\Gamma) = M_{A_p}(\Gamma).$$

Since $M_{A_p}(\Gamma) = (M(G))^{\wedge}$ provided $1 \leq p < \infty$ and G is a non-compact locally compact abelian group and $M_{A_p}(\Gamma) = C(\Gamma)$ when G is infinite compact abelian and $1 \leq p \leq 2$, therefore we have the following two corollaries.

COROLLARY 4.2.2 : Let G be a noncompact locally compact abelian group and $1 \leq p \leq q < \infty$, then

$$M_{A_p}^{L_1}(\Gamma) = M_{A_p}^{A_q}(\Gamma) = (M(G))^{\wedge}$$

COROLLARY 4.2.3 : Let G be an infinite compact abelian group and $1 \leq p \leq 2$, $p \leq q < \infty$, then

$$M_{A_p}^{L_1}(\Gamma) = M_{A_p}^{A_q}(\Gamma) = C(\Gamma)$$

The following theorem and corollary are due to Krogstad [14], although partially or fully these results are also proved by number of other authors. See for example [3], [4], [22].

THEOREM 4.2.4 : Let G be a nondiscrete locally compact abelian group. If $1 \leq q < \infty$ and $T: L_1(G) \to A_q(G)$, then the following are equivalent:

(i) $T \in M(L_1, A_q)$

(ii) There exists a unique $\mu \in B_q(G)$ such that $Tf = \mu * f$, $f \in L_1(G)$. Moreover the mapping $\mu \to T$ from $B_q(G)$ to $M(L_1, A_q)$

is an isometric surjective algebra isomorphism.

COROLLARY 4.2.5 : Let G be a nondiscrete locally compact abelian group. If $1 \leq q \leq 2$ and $T: L_1(G) \to A_q(G)$, then the following are equivalent.

(i) $T \in M(L_1, A_q)$

(ii) There exists a unique $g \in A_q(G)$ such that $Tf = g * f$, $f \in L_1(G)$.

Moreover the mapping $g \to T$ from $A_q(G)$ to $M(L_1, A_q)$ is an isometric surjective algebra isomorphism.

REMARK 4.2.6 : Krogstad in [14] showed that if G is a non-discrete locally compact abelian group and $q > 2$, then there exists a $\mu \in M(G)$ such that $\hat{\mu} \in L_q(\Gamma)$ but μ is singular with respect to Haar measure on G. This shows that $M_{L_1}^{A_q}(\Gamma) \neq (A_q(G))^\wedge$ provided $q > 2$.

Let us now focus our attention on studying $M_{A_p}^{A_q}(\Gamma)$ when $1 \leq q < p < \infty$.

PROPOSITION 4.2.7: Let G be a non-discrete non-compact locally compact abelian group. Then

$$B_{pq/p-q}(G)^\wedge \subseteq M_{A_p}^{A_q}(\Gamma) \subsetneq (M(G))^\wedge$$

PROOF : Let $\phi = \hat{\mu}$, $\mu \in B_{pq/p-q}(G)$ and $f \in A_p(G)$. Then $\phi \hat{f} = \hat{\mu} \hat{f} = (\mu * f)^\wedge \in (L_1(G))^\wedge$. Moreover, by Hölder's inequality we have

$$\left(\int_{\Gamma} |\phi \hat{f}(\gamma)|^{q} \, dn(\gamma) \right)^{1/q} \leq \left(\int_{\Gamma} |\phi(\gamma)|^{pq/p-q} dn(\gamma) \right)^{p-q/p} \left(\int_{\Gamma} |\hat{f}(\gamma)|^{p} dn(\gamma) \right)^{1/p}$$

$$< \infty$$

Hence $\phi \hat{f} \in (A_q(G))^{\wedge}$ and $\phi \in M_{A_p}^{A_q}(\Gamma)$. Consequently,

$$(B_{pq/p-q}(G))^{\wedge} \subseteq M_{A_p}^{A_q}(\Gamma).$$

Evidently $M_{A_p}^{A_q}(\Gamma) \subseteq M_{A_p}(\Gamma) = M(G)^{\wedge}$. $M_{A_p}^{A_q}(\Gamma) \neq M(G)^{\wedge}$ follows by Theorem 3.1.13.

Now let G be an infinite compact abelian group. The following proposition gives $M_{A_p}^{A_q}(\Gamma)$ provided $p \leq 2$.

PROPOSITION 4.2.8 : Let G be an infinite compact abelian group and $1 \leq q < p \leq 2$, then

$$M_{A_p}^{A_q}(\Gamma) = \ell_{pq/p-q}(\Gamma) .$$

PROOF : We observe that if $1 \leq r \leq 2$, then $\ell_r(\Gamma) = (A_r(G))^{\wedge}$. By Theorem 2.4.3 we have

$$\ell_{pq/p-q}(\Gamma) = [\ell_p, \ell_q] = [A_p, A_q] = M_{A_p}^{A_q}(\Gamma).$$

PROPOSITION 4.2.9 : Let G be an infinite compact abelian group and $1 \leq q \leq 2 < p$. Then

$$\ell_{pq/p-q}(\Gamma) \subseteq M_{A_p}^{A_q}(\Gamma) \subsetneqq \ell_{2q/2-q}(\Gamma).$$

Moreover, if $r > \frac{pq}{p-q}$, then $\ell_r(\Gamma) \not\subseteq M_{A_p}^{A_q}(\Gamma)$ and if in addition

$p \leq 2q$, then $\ell_{pq/p-q}(\Gamma) \not\subseteq M_{A_r}^{A^q}(\Gamma)$.

PROOF : It is clear by Hölder's inequality and Proposition 4.2.8 that $\ell_{pq/p-q}(\Gamma) \subseteq M_{A_p}^{A^q}(\Gamma) = \ell_{2q/2-q}(\Gamma)$, as $M_{A_p}^{A^q}(\Gamma) \subseteq M_{A_2}^{A^q}(\Gamma)$. So let $r > \frac{pq}{p-q}$, then $p > \frac{rq}{r-q}$. By Theorem 3.1.8 there exists a $f \in A_p(G)$ such that $\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^{rq/r-q} = \infty$. Hence by Theorem 2.4.3 there exists a function $\psi \in \ell_r(\Gamma)$ such that $\psi \hat{f} \notin \ell_q(\Gamma)$. Consequently, $\ell_r(\Gamma) \not\subseteq M_{A_p}^{A^q}(\Gamma)$. We are now to prove that $\ell_{pq/p-q}(\Gamma) \not\subseteq M_{A_r}^{A^q}(\Gamma)$ provided $p \leq 2q$. It is clear that $r > \frac{pq}{p-q} \geq p$, provided $p \leq 2q$. By Theorem 3.1.8 choose a function $f \in A_r(G)$ such that $\sum_{\gamma} |\hat{f}(\gamma)|^p = \infty$. Then by Theorem 2.4.3 there exists a function $\psi \in \ell_{pq/p-q}(\Gamma)$ such that $\psi \hat{f} \notin \ell_q(\Gamma)$. Thus $\psi \notin M_{A_r}^{A^q}(\Gamma)$.

Let us now consider the case when $2 < q < p < \infty$. In this situation $\ell_{pq/p-q}(\Gamma)$ need not be contained in $M_{A_p}^{A^q}(\Gamma)$. The following proposition tells us that when $4 < p \leq 2q$ or $4 < q < p < \infty$, then $\ell_{pq/p-q}(\Gamma)$ ceases to be contained in $M_{A_p}^{A^q}(\Gamma)$.

PROPOSITION 4.2.10 : Let G be an infinite compact abelian group and either $4 < q < p < \infty$ or $4 < p \leq 2q$, then

$$\ell_{pq/p-q}(\Gamma) \not\subseteq M_{A_p}^{A^q}(\Gamma) \text{ if } p > q.$$

Before proving this proposition we shall first prove a lemma.

LEMMA 4.2.11 : Let G be an infinite compact abelian group. If $1 \leq p \leq 4$, then $\ell_p(\Gamma) \subseteq M_{A_p}(\Gamma)$, but if $p > 4$ then $\ell_4(\Gamma) \not\subseteq M_{A_p}(\Gamma)$ and $\ell_p(\Gamma) \not\subseteq M_{A_4}(\Gamma)$.

PROOF : It is clear from corollary 4.2.3 that we can assume $p > 2$. Therefore suppose that $2 < p \leq 4$. Evidently $\frac{2p}{p-2} \geq p$. Now it follows from (4.1.1b) that $\ell_p(\Gamma) \subseteq M_{A_p}(\Gamma)$.

Let us now consider the case when $4 < p < \infty$. Then it follows from (4.1.1a) that $\ell_4(\Gamma) \not\subseteq M_{A_p}(\Gamma)$. Now to prove that $\ell_p(\Gamma) \not\subseteq M_{A_4}(\Gamma)$, we observe that $\frac{2p}{p-2} < 4$. Again by choosing a function $f \in A_4(G)$ such that $f \notin A_{2p/p-2}(G)$ and proceeding as in Proposition 4.1.1 it can be easily seen that $\ell_p(\Gamma) \not\subseteq M_{A_4}(\Gamma)$.

PROOF OF PROPOSITION 4.2.10 : If $4 < q < p < \infty$ or $4 < p \leq 2q$, then $\frac{pq}{p-q} \geq 4$ provided $p > q$. Now by Lemma 4.2.11 it follows that $\ell_{pq/p-q}(\Gamma) \not\subseteq M_{A_p}^{A_q}(\Gamma)$ as $M_{A_p}(\Gamma) \supseteq M_{A_p}^{A_q}(\Gamma)$.

Propositions 4.2.8 to 4.2.10 tell us that $\ell_{pq/p-q}(\Gamma) \subseteq M_{A_p}^{A_q}(\Gamma)$ provided $p > q$, $1 \leq q \leq 2$; however $\ell_{pq/p-q}(\Gamma)$ need not be contained in $M_{A_p}^{A_q}(\Gamma)$ if $2 < q < p < \infty$. The following proposition tells

us that for $2 < q < p < \infty$ $M_{A_p}^{A_q}(\Gamma)$ properly contains $(B_{pq/p-q}(G))^{\wedge}$.

PROPOSITION 4.2.12: Let G be an infinite compact abelian group and $2 < q < p < \infty$, then

$$(B_{pq/p-q}(G))^{\wedge} \subsetneqq M_{A_p}^{A_q}(\Gamma).$$

PROOF : It is easy to see that $(B_{pq/p-q}(G))^{\wedge} \subseteq M_{A_p}^{A_q}(\Gamma)$. We are now to prove that the inclusion is proper. By Theorem 3.1.8 choose a function $f \in A_{2p/p-2}(G)$ such that $\hat{f} \notin \ell_2(\Gamma)$. Therefore by Theorem 2.4.6 there exists a function $\varepsilon(\gamma) = \pm 1$ on Γ such that $\varepsilon(\gamma)\hat{f}(\gamma)$ is not the Fourier-Stieltjes transform of any measure on G. However $\varepsilon(\gamma)\hat{f}(\gamma) \in \ell_{2p/p-2}(\Gamma)$ and it can be easily seen that $\ell_{2p/p-2}(\Gamma) \subseteq M_{A_p}^{A_2}(\Gamma) \subseteq M_{A_p}^{A_q}(\Gamma)$.

REMARK 4.2.13 : It can be easily seen that Proposition 4.2.1 and Corollary 4.2.3 are valid for infinite compact groups. Theorem 4.2.4 was proved by Krogstad [14] . But his proof does not seem to work when group G is infinite compact. We shall now give the proof of Theorem 4.2.4 in this situation.

PROPOSITION 4.2.14 : Let G be an infinite compact group and $1 \leq q < \infty$. Let $\mu \in M(G)$ be such that $\mu * L_1(G) \subseteq A_q(G)$, then $\hat{\mu} \in C_q(\Sigma)$.

PROOF : Suppose $\hat{\mu} \notin C_q(\Sigma)$. Choose a sequence $\{\psi_n\}_{n=1}^{\infty}$ of finite subsets of Σ such that

$$\sum_{\sigma \in \psi_n} d_\sigma \|\hat{\mu}(\sigma)\|_{\phi_q}^q \geq n^{3q}$$

Now by [12, 28.53] choose a sequence $\{h_k\}$ in $L_1(G)$ such that $\|h_n\|_1 = 1$, $\hat{h}_n(\sigma) = \alpha_n(\sigma) I_{d_\sigma}$, $\alpha_n(\sigma) \geq 0$, $\alpha_n(\sigma) > \frac{1}{2} \forall \sigma \in \psi_n$.

Let $h = \sum_{n=1}^{\infty} \frac{h_n}{n^2}$; then $h \in L_1(G)$, $\hat{h}(\sigma) = \alpha(\sigma) I_{d_\sigma}$ and $\alpha(\sigma) \geq 0 \ \forall \sigma \in \Sigma$ and $\alpha(\sigma) \geq \frac{1}{2n^2} \forall \sigma \in \psi_n$. Now

$$\sum_{\sigma} d_\sigma \|\hat{\mu}(\sigma)\, \hat{h}(\sigma)\|_{\phi_q}^q$$

$$= \sum_{\sigma} d_\sigma (\alpha(\sigma))^q \|\hat{\mu}(\sigma)\|_{\phi_q}^q$$

$$\geq \sum_{\sigma \in \psi_n} d_\sigma \frac{1}{2^q n^{2q}} \|\hat{\mu}(\sigma)\|_{\phi_q}^q$$

$$\geq \frac{1}{2^q} \frac{n^{3q}}{n^{2q}} = (n/2)^q \ \forall n, \text{ a contradiction.}$$

PROPOSITION 4.2.15: Let G be an infinite compact group and $1 \leq q \leq 2$. Then

$$M_{L_1}^{A_q}(\Sigma) = (A_q(G))^\wedge$$

PROOF : It is obvious that $(A_q(G))^\wedge \subseteq M_{L_1}^{A_q}(\Sigma)$. Conversely, $M_{L_1}^{A_q}(\Sigma) \subseteq M_{L_1}^{L_1}(\Sigma) = M(G)^\wedge$. Hence if $\hat{\mu} \in M_{L_1}^{A_q}(\Sigma)$, then

$\mu * L_1(G) \subseteq A_q(G)$. Therefore by Proposition 4.2.14 $\hat{\mu} \in C_q(\Sigma)$. Then it follows from [12, 34.47(b)] that $\mu \in A_q(G)$.

REMARK 4.2.16 : Propositions 4.2.8 to 4.2.10 and Proposition 4.2.12 are true for infinite compact groups. The arguments used in the proofs of these Propositions are valid in this situation. See for example, Theorem 3.1.10, Remark 2.4.9.

4.3 RESTRICTION AND EXTENSION PROBLEMS FOR $M_{A_p}^{A_q}(\Gamma)$: Let Γ_o be a closed subgroup of Γ. It is known [29, Remark (b)] that restrictions to Γ_o of the functions in $M_{L_p}(\Gamma) \cap C(\Gamma)$ belong to $M_{L_p}(\Gamma_o) \cap C(\Gamma_o)$. Conversely, it is not known if every function in $M_{L_p}(\Gamma_o) \cap C(\Gamma_o)$ is the restriction of a function in $M_{L_p}(\Gamma) \cap C(\Gamma)$; this is the case if $p = 1$ or 2 or Γ_o is a closed discrete subgroup of Γ, see [10]. The author of [29] conjectures that this is true for all p with $1 < p < 2$. In this section we answer the corresponding problems for A_p-multipliers. The following proposition tells us that if $\phi \in M_{A_p}(\Gamma)$ and $\psi = \phi|_{\Gamma_o}$, then $\psi \in M_{A_p}(\Gamma_o)$.

PROPOSITION 4.3.1 : Let G be a locally compact abelian group. If $\phi \in M_{A_p}(\Gamma)$, $1 \leq p < \infty$, then $\phi|_{\Gamma_o} \in M_{A_p}(\Gamma_o)$.

PROOF : Case I : G noncompact – In this situation $M_{A_p}(\Gamma) = (M(G))^\wedge = B(\Gamma)$. But by Theorem 2.7.2 of [28] functions in

$B(\Gamma_o)$ are precisely restrictions of functions in $B(\Gamma)$ to Γ_o.

Therefore,

$$\phi|\Gamma_o = \hat{\mu}|\Gamma_o \in B(\Gamma_o) \subseteq M_{A_p}(\Gamma_o) .$$

Case II : G compact : Let H be the annihilator of Γ_o in G. Then H is compact. We shall show that $\phi|_{\Gamma_o} = \psi \in M_{A_p}(\Gamma_o)$.

Let $f \in A_p(G/H)$, then by Proposition 3.1.6 there exists $f' \in A_p(G)$ such that

$$(4.3.1a) \qquad \Pi_H(f') = f .$$

But $\phi \in M_{A_p}(\Gamma)$ and $f' \in A_p(G)$, therefore there exists $g' \in A_p(G)$ such that

$$(4.3.1b) \qquad \phi \hat{f}' = \hat{g}' .$$

Let $g = \Pi_H(g')$, then $g \in A_p(G/H)$ and

$$\begin{aligned} \psi\hat{f} = \phi|_{\Gamma_o} \hat{f} &= \phi|_{\Gamma_c} (\Pi_H(f'))\hat{} \quad \text{(cf, 4.3.1a)} \\ &= \phi|_{\Gamma_o} \hat{f}'|_{\Gamma_o} = \phi\hat{f}'|_{\Gamma_o} \\ &= \hat{g}'|_{\Gamma_o} \quad \text{(cf. 4.3.1b)} \\ &= \Pi_H(g')\hat{} = \hat{g} . \end{aligned}$$

Therefore $\psi \in M_{A_p}(\Gamma_o)$.

Proposition 4.3.1 shows that if Γ_o is a closed subgroup of Γ , then

$$M_{A_p}(\Gamma)|_{\Gamma_0} \subseteq M_{A_p}(\Gamma_0).$$

Now we ask, is $M_{A_p}(\Gamma)|_{\Gamma_0} = M_{A_p}(\Gamma_0)$? That is, whether every function $\phi \in M_{A_p}(\Gamma_0)$ is the restriction of some function in $M_{A_p}(\Gamma)$. One can easily see that it is not always true. Indeed, let $G = R$ and $\Gamma_0 = Z$, then $M_{A_p}(\Gamma) = M(G)^\wedge$, but $M_{A_p}(\Gamma_0)$ properly contains measures [17, pp 207-208] .

The following proposition tells when the extension is possible for noncompact G. If G is an infinite compact abelian group, the question is answered in Proposition 4.3.3.

PROPOSITION 4.3.2 : Let G be a noncompact locally compact abelian group and Γ_0 be a nondiscrete closed subgroup of Γ . If $\phi \in M_{A_p}(\Gamma_0)$, then there exists a $\psi \in M_{A_p}(\Gamma)$ such that $\phi = \psi|_{\Gamma_0}$.

PROOF: We have $M_{A_p}(\Gamma) = (M(G))^\wedge = B(\Gamma)$. Also, $M_{A_p}(\Gamma_0)$ is the set of multipliers of $A_p(G/H)$, where H is the annihilator of Γ_0 in G. Since Γ_0 is nondiscrete, G/H is noncompact. Thus $M_{A_p}(\Gamma_0) = B(\Gamma_0)$. But by Theorem 2.7.2 of [28] functions in $B(\Gamma_0)$ are precisely restrictions of functions in $B(\Gamma)$. Hence the result.

PROPOSITION 4.3.3 : Let G be an infinite compact abelian group and Γ_0 be a subgroup of Γ . If $\phi \in M_{A_p}^{A^q}(\Gamma_0)$, $1 \leq p,q < \infty$, then there exists a $\psi \in M_{A_p}^{A^q}(\Gamma)$ such that $\phi = \psi|_{\Gamma_0}$.

PROOF : Let H be the annihilator of Γ_o in G. Since Γ_o is discrete, H is compact. Let $\phi \in M_{A_p}^{A_q}(\Gamma_o)$. Define ψ on Γ as follows

$$\psi(\gamma) = \begin{cases} \phi(\gamma) & \gamma \in \Gamma_o \\ 0 & \gamma \notin \Gamma_o \end{cases}$$

We shall show that $\psi \in M_{A_p}^{A_q}(\Gamma)$. Let $f \in A_p(G)$, then by Proposition 3.1.6 $\Pi_H(f) \in A_p(G/H)$. But $\phi \in M_{A_p}^{A_q}(\Gamma_o)$, therefore there exists $g' \in A_q(G/H)$ such that

$$(4.3.3a) \qquad \phi\,(\Pi_H(f))^\wedge = \hat{g}'.$$

Now by Proposition 3.1.6 there exists $h' \in A_q(G)$ such that $\Pi_H(h') = g'$. Let $h = m_H * h'$, where m_H is the normalised Haar measure on H and m_H is considered as a bounded measure on the whole group G in the usual manner. Then,

$$(4.3.3b) \qquad \psi \hat{f} = \hat{h}$$

as can be seen from the following.

If $\gamma \in \Gamma_o$, then

$$\begin{aligned} \psi\hat{f}(\gamma) = \phi(\gamma)\;(\Pi_H(f))^\wedge(\gamma) &= \hat{g}'(\gamma) \quad (\text{cf } 4.3.3\text{ a}) \\ &= (\Pi_H(h'))^\wedge(\gamma) \\ &= \hat{h}'(\gamma) = \hat{h}(\gamma). \end{aligned}$$

If $\gamma \notin \Gamma_0$, then $\psi(\gamma)\,\hat{f}(\gamma) = 0 = \hat{h}\,(\gamma)$. Thus, $\psi\,\hat{f}(\gamma) = \hat{h}(\gamma)$ $\forall\, \gamma \in \Gamma$. Consequently $\psi \in M_{A_p}^{q}{}^{A}(\Gamma)$.

Let G be an infinite compact abelian group; then proceeding as in Proposition 4.3.1 one can show that $M_{A_p}^{q}{}^{A}(\Gamma)|_{\Gamma_0} \subseteq M_{A_p}^{q}{}^{A}(\Gamma_0)$, $1 \leq p,q < \infty$. This fact and Proposition 4.3.3 give the following corollary.

COROLLARY 4.3.4 : Let G be an infinite compact abelian group and Γ_0 a subgroup of Γ, $1 \leq p,q < \infty$. Then $M_{A_p}^{q}{}^{A}(\Gamma)|_{\Gamma_0} = M_{A_p}^{q}{}^{A}(\Gamma_0)$.

Let G be an infinite compact group and H a closed normal subgroup of G. Let $[P]$ be the annihilator of H in Σ ; then the following analogues of Proposition 3.1.6 and Corollary 4.3.4 are valid in this situation.

PROPOSITION 4.3.5: Let $1 \leq p < \infty$, then $\Pi_H(A_p(G)) = A_p(G/H)$.

PROPOSITION 4.3.6 : Let $1 \leq p,q < \infty$, then $M_{A_p}^{q}{}^{A}(\Sigma)|_{[P]} = M_{A_p}^{q}{}^{A}([P])$.

CHAPTER - V

CONTINUOUS FUNCTIONS WITH P-SUMMABLE FOURIER TRANSFORMS

In chapters III and IV we studied integrable functions on locally compact abelian or infinite compact groups with p summable Fourier transforms. In this chapter we study another important class of functions on infinite compact abelian groups, namely, $E_p(G) = \{f \in C(G) \mid \hat{f} \in \ell_p(\Gamma)\}$. Since every continuous function on G has Fourier transform in $\ell_2(\Gamma)$, therefore $E_p(G) = E_2(G) = C(G)$ if $p \geq 2$. Thus interest lies in the case when $1 \leq p \leq 2$. Henceforth, throughout this chapter, we have $1 \leq p, q \leq 2$ and G an infinite compact abelian group unless otherwise stated.

Lynette in [23] showed that if we take $G = T$, $1 \leq p < 2$ and pointwise multiplication in $C(G)$, then $E_p(G)$ is not closed under this operation; in other words, $E_p(G)$ is not an algebra. But we shall show that if we take convolution as multiplication in $C(G)$ and define norm of an element $f \in E_p(G)$ as $||f||_{E_p} = ||f||_\infty + ||\hat{f}||_p$, then $E_p(G)$ is a commutative semisimple Banach algebra.

In section 1 of this chapter we shall show that $E_p(G)$ is a Segal algebra on G and $\bigcup_{p<q} E_p(G)$ is a $|| \ ||_{E_q}$ dense subspace of $E_q(G)$ of category I. In Section 2 we improve some results of Lynette [24] concerning multipliers from $E_p(G)$ to $E_q(G)$ algebras.

Section 3 deals with restriction of $M_{E_p}^{E_q}(\Gamma)$ to closed subgroups of Γ and the corresponding extension problem. Section 4 deals with multipliers between $L_1(G)$ and $E_q(G)$ and it is shown that if $1 < q \leq 2$ and $G = T$, then $E_q(G)$ is not dual of an L_1-module. Section 5 deals with multipliers between $A_p(G)$ and $E_q(G)$ algebras, $1 \leq p < \infty$.

5.1 SOME BASIC RESULTS ON $E_p(G)$ ALGEBRAS : Since $C(G)$ is an ideal in $L_1(G)$, it is evident that $E_p(G)$ is an ideal in $L_1(G)$. It is easy to see that $||f||_{E_p} = ||f||_\infty + ||\hat{f}||_p$, $f \in E_p(G)$, defines a norm on $E_p(G)$ and that $E_p(G)$ is a commutative algebra with convolution as multiplication. The following proposition states that $E_p(G)$ is also a Banach algebra.

PROPOSITION 5.1.1 : $E_p(G)$ is a commutative Banach algebra with $||\ \ ||_{E_p}$ and convolution as multiplication.

PROOF : Let $\{f_n\}$ be a Cauchy sequence in $||\ \ ||_{E_p}$. Clearly $\{f_n\}$ and $\{\hat{f}_n\}$ are Cauchy sequences in $C(G)$ and $\ell_p(\Gamma)$ respectively. Let $f \in C(G)$ and $g \in \ell_p(\Gamma)$ be such that $\lim_n ||f_n - f||_\infty = 0$ and $\lim_n ||\hat{f}_n - g||_p = 0$. The first assertion implies that $\lim_n ||\hat{f}_n - \hat{f}||_\infty = 0$. Thus $\hat{f} = g$. Hence $f \in E_p(G)$ and $\lim_n ||f_n - f||_{E_p} = 0$. Therefore $E_p(G)$ is a complete space with respect to $||\ \ ||_{E_p}$. As indicated previously $E_p(G)$ is a commutative algebra. To show $E_p(G)$ is a Banach algebra, we observe that for each $f,g \in E_p(G)$,

$$\begin{aligned}||f * g||_{E_p} &= ||f * g||_\infty + ||\hat{f}\,\hat{g}||_p \\ &\le ||f||_\infty\,(||g||_\infty + ||\hat{g}||_p) \\ &\le ||f||_{E_p}\,||g||_{E_p}.\end{aligned}$$

PROPOSITION 5.1.2 : $E_p(G)$ is a Segal algebra.

PROOF : It is evident that $E_p(G)$ is translation invariant and $||f_x||_{E_p} = ||f||_{E_p}$, $f \in E_p(G)$ and $x \in G$. Clearly $E_p(G)$ is norm dense in $L_1(G)$ and we have already shown that $E_p(G)$ is a Banach algebra. We are now to show that the mapping $x \to f_x$ from G into $E_p(G)$ is continuous. Let $0 \neq f \in E_p(G)$ and $\varepsilon > 0$. First choose a neighbourhood U of the identity element 0 of G such that

(5.1.2a) $\qquad ||f_x - f||_\infty < \varepsilon/2$ for all $x \in U$.

Choose a function ϕ on Γ having finite support such that $||\phi - \hat{f}||_p < \varepsilon/8$. Let K be the support of ϕ and $K' = \Gamma \setminus K$, then it follows that

(5.1.2b) $\qquad ||\hat{f}\,\chi_{K'}||_p < \varepsilon/8$.

Now define,

$$N(K,\varepsilon) = \{x \in G \mid |(x,\gamma) - 1| < \frac{\varepsilon}{4\,||\hat{f}||_p} \text{ for all } \gamma \in K\}.$$

Then $N(K,\varepsilon)$ is a neighbourhood of 0 in G. We now choose a symmetric neighbourhood W of 0 such that $W \subset U \cap N(K,\varepsilon)$.

It follows that

(i) for $x \in W$ and $\gamma \in K$ we have

$$|\hat{f}_x(\gamma) - \hat{f}(\gamma)| = |(-x,\gamma) - 1| \; |\hat{f}(\gamma)|$$

$$< \frac{\varepsilon}{4||\hat{f}||_p} |\hat{f}(\gamma)|,$$

and hence we have

$$||(\hat{f}_x(\gamma) - \hat{f}(\gamma)) \; \chi_K||_p < \varepsilon/4 \;;$$

(ii) for $x \in W$ and $\gamma \in K'$ we have

$$|\hat{f}_x(\gamma) - \hat{f}(\gamma)| \leq 2 \, |\hat{f}(\gamma)| .$$

Then it follows from (5.1.2b) that

$$||(\hat{f}_x - \hat{f}) \, \chi_{K'} \, ||_p < \varepsilon/4 .$$

Thus for $x \in W$ we have

$$||\hat{f}_x - \hat{f}||_p < \varepsilon/2 \quad \text{and hence}$$

$$||f_x - f||_{E_p} = ||f_x - f||_\infty + ||(f_x - f)^\wedge||_p < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Evidently $E_p(G) \subseteq E_q(G)$ if $p < q$. For $G = T$, the circle group, Lynette in [24] shows that $E_p(G) \subsetneq E_q(G)$. However, in the following proposition we shall derive $E_p(G) \subsetneq E_q(G)$ as an application of Theorem 3.1.8.

PROPOSITION 5.1.3 : Let $p < q$, then $E_p(G)$ is a $|| \; ||_{E_q}$ dense linear subspace of $E_q(G)$ of category I.

PROOF : Since integrable functions on G whose Fourier transforms

have finite support are dense in $E_q(G)$, therefore $E_p(G)$ is a $\|\ \|_{E_q}$ dense linear subspace of $E_q(G)$. To show that $E_p(G)$ is of category I in $E_q(G)$, it is enough to show that $E_p(G) \subsetneq E_q(G)$. It is clear that $p < q$ implies $\frac{2p}{2-p} < \frac{2q}{2-q}$. Hence by Theorem 3.1.8 there exists a function $f \in A_{2q/2-q}(G)$, but $\hat{f} \notin \ell_{2p/2-p}(\Gamma)$. Then by Theorem 2.4.8 there exists a function $g \in C(G)$ such that $\hat{g}\hat{f} \notin \ell_p(\Gamma)$. Let $h = g * f$, then h is a continuous function on G such that $\hat{h} \notin \ell_p(\Gamma)$. However $\hat{h} \in \ell_q(\Gamma)$ (because $\hat{g} \in \ell_2(\Gamma)$ and $\hat{f} \in \ell_{2q/2-q}(\Gamma)$).

COROLLARY 5.1.4 : $\bigcup_{p<q} E_p(G)$ is a $\|\ \|_{E_q}$ dense linear subspace of $E_q(G)$ of category I.

In Chapter III it was proved in Corollary 3.1.14 that $A_p(G)$, $1 \leq p < \infty$, is not a prime ideal in $L_1(G)$. It is natural to ask if $E_p(G)$ is a prime ideal in $L_1(G)$. The following corollary answers this question.

COROLLARY 5.1.5 : $E_p(G)$ is not a prime ideal in $L_1(G)$.

PROOF : Let $1 \leq p < 2$. By Proposition 5.1.3 choose a $f \in C(G)$ such that $f \notin E_p(G)$. Then $f * f \in E_1(G) \subseteq E_p(G)$. If $p = 2$, then by choosing a $f \in L_\infty(G)$ such that $f \notin C(G)$ it follows that $E_2(G)$ is not a prime ideal in $L_1(G)$.

REMARK 5.1.6 : In view of Remark 2.4.9 and Theorem 3.1.10, it is easy to see that results proved in this section are true for infinite compact groups.

5.2 MULTIPLIERS BETWEEN $E_p(G)$ AND $E_q(G)$ ALGEBRAS.

Lynette in [24] has studied $M(E_p, E_q)$. Although she has obtained results for $G = T$, the circle group, but the following results of hers are valid for any infinite compact abelian group. We are giving these results as a pretext for studying multipliers between $A_p(G)$ $(1 \leq p < \infty)$ and $E_q(G)$ algebras.

PROPOSITION 5.2.1 : Let $p \leq q$, then

$$M(E_p) = M(E_p, E_q) = M(E_p, C) = M(G) + PM^{p'}$$

where $PM^{p'}$ denotes the pseudomeasures on G whose Fourier transforms belong to $\ell_{p'}(\Gamma)$.

PROPOSITION 5.2.2 : Let $q < p$, then

(i) $(B_{pq/p-q}(G))^{\wedge} \subseteq M_{E_p}^{E_q}(\Gamma)$

(ii) $M_{E_2}^{E_q}(\Gamma) \subseteq \ell_{2q/2-q}(\Gamma)$

(iii) $M_{E_2}^{E_1}(\Gamma) = \ell_2(\Gamma)$.

Proposition 5.2.2 can be improved. The improvement is the following.

PROPOSITION 5.2.3 : Let $q < p$, then

(i) $(B_{pq/p-q}(G))^{\wedge} \subsetneq M_{E_p}^{E_q}(\Gamma)$

(ii) $\ell_{p'}(\Gamma) \subseteq M_{E_p}^{E_q}(\Gamma)$ and if $q > 1$, then inclusion is proper.

Moreover, $\ell_{p'}(\Gamma)$ is the largest $\ell_r(\Gamma)$ space contained in $M_{E_p}^{E_q}(\Gamma)$.

(iii) $M_{E_2}^{E\,q}(\Gamma) \subsetneqq \ell_{2q/2-q}(\Gamma)$ if $q > 1$.

PROOF : (i) Choose a function $\phi \in \ell_{p'}(\Gamma)$ such that $\phi \notin \ell_2(\Gamma)$. Then by Theorem 2.4.6 there exists a function $\varepsilon(\gamma) = \pm 1$ on Γ such that $\varepsilon(\gamma)\phi(\gamma)$ is not the Fourier-Stieltjes transform of any measure on G. However $\psi(\gamma) = \varepsilon(\gamma)\phi(\gamma) \in \ell_{p'}(\Gamma)$ and $\ell_{p'}(\Gamma) \subseteq M_{E_p}^{E\,1}(\Gamma) \subseteq M_{E_p}^{E\,q}(\Gamma)$.

(ii) It is clear that $\ell_{p'}(\Gamma) \subseteq M_{E_p}^{E\,q}(\Gamma)$. Now to show that this inclusion is proper, we observe that if $q > 1$, then $\frac{pq}{p-q} > p'$. Then it follows from Theorem 3.1.8 that $\ell_{p'}(\Gamma) \subsetneqq M_{E_p}^{E\,q}(\Gamma)$. Let $r > p'$; then $p > r'$. By Theorem 3.1.8 choose a function $f \in A_p(G)$ such that $\hat{f} \notin \ell_{r'}(\Gamma)$. Theorem 2.4.3 then implies that there exists a function $\phi \in \ell_r(\Gamma)$ such that $\phi\hat{f} \notin \ell_1(\Gamma)$. Now by Theorem 2.4.7 there exists a function $\varepsilon(\gamma) = \pm 1$ on Γ such that $\varepsilon(\gamma)\phi(\gamma)\hat{f}(\gamma)$ is not the Fourier transform of any continuous function. Now it is evident that $\varepsilon(\gamma)\phi(\gamma) = \psi(\gamma)$ is a function in $\ell_r(\Gamma)$ but $\psi \notin M_{E_p}^{E\,q}(\Gamma)$.

(iii) Follows from (ii).

REMARK 5.2.4 : It is easy to see that Proposition 5.2.3 is true for infinite compact groups except the latter assertion in (ii), because we are not aware whether Theorem 2.4.7 has an analogue for infinite compact groups; for the other arguments used in Proposition 5.2.3, see Theorem 3.1.10 and Remark 2.4.9.

5.3 RESTRICTION AND EXTENSION OF $E_p(G)$ TO $E_q(G)$ MULTIPLIERS :

In Chapter IV we studied the restriction of multipliers of $A_p(G)$ to closed subgroups Γ_o of Γ and the corresponding extension problem. We showed that if Γ_o is a subgroup of Γ, then $M_{A_p}^{A_q}(\Gamma)|\Gamma_o = M_{A_p}^{A_q}(\Gamma_o)$, $1 \leq p,q < \infty$. In this section we shall show that $M_{E_p}^{E_q}(\Gamma)|_{\Gamma_o} = M_{E_p}^{E_q}(\Gamma_o)$.

THEOREM 5.3.1 : Let Γ_o be a subgroup of Γ. Then $M_{E_p}^{E_q}(\Gamma)|\Gamma_o = M_{E_p}^{E_q}(\Gamma_o)$.

Before proving this theorem we shall prove a lemma.

LEMMA 5.3.2 : Let Γ_o be a subgroup of Γ and H the annhilator of Γ_o in G. Then, $\Pi_H(E_p(G)) = E_p(G/H)$.

PROOF : It is clear that H is a compact subgroup of G. Now $\Pi_H(E_p(G)) = \Pi_H(C(G) \cap A_p(G)) \subseteq \Pi_H(C(G)) \cap \Pi_H(A_p(G)) \subseteq C(G/H) \cap A_p(G/H) = E_p(G/H)$. To prove equality let $f' \in E_p(G/H)$. Choose $f \in C(G)$ such that $\Pi_H(f) = f'$. Let $g = m_H * f$ as in Proposition 3.1.6. It is clear that $g \in C(G)$ and as in Proposition 3.1.6, $g \in E_p(G)$ and $\Pi_H(g) = f'$.

PROOF OF THEOREM 5.3.1 : Proof of this theorem is similar to that of Corollary 4.3.4 and follows by replacing $A_p(G)$, $A_q(G)$ by $E_p(G)$, $E_q(G)$ and using Lemma 5.3.2.

Let G be an infinite compact group and H a closed normal subgroup of G. Let $[P]$ be the annihilator of H in Σ ; then the following analogues of Lemma 5.3.2 and Theorem 5.3.1 are valid in this situation.

PROPOSITION 5.3.3 : $\pi_H(E_p(G)) = E_p(G/H)$

PROPOSITION 5.3.4 : $M_{E_p}^{E_q}(\Sigma)|_{[P]} = M_{E_p}^{E_q}([P])$

5.4 MULTIPLIERS BETWEEN $L_1(G)$ AND $E_q(G)$: It can be shown using Burnham's result in [3] that $M(L_1,E_q) = \{f \in L_\infty(G) | \hat{f} \in \ell_q(\Gamma)\}$.
However, without using Burnham's result we shall give a simple proof of this result.

PROPOSITION 5.4.1 : $M(L_1,E_q) = \{f \in L_\infty(G) \mid \hat{f} \in \ell_q(\Gamma)\}$.

PROOF : Let $f \in L_\infty(G)$ such that $\hat{f} \in \ell_q(\Gamma)$. Then the mapping $T: L_1(G) \to E_q(G)$ defined by $T(g) = f * g$ is a continuous linear transformation commuting with convolution. Therefore it follows that

$$\{f \in L_\infty(G) \mid \hat{f} \in \ell_q(\Gamma) \subseteq M(L_1,E_q)$$

Conversely, let $T \in M(L_1,E_q) \subseteq M(L_1,A_q)$. Therefore it follows from Corollary 4.2.5 that there exists $g \in A_q(G)$ such that $Tf = f * g \; \forall f \in L_1(G)$ and there exists a positive constant $K > 0$ such that

(5.4.1a) $\quad \|Tf\|_{E_q} = \|f * g\|_{E_q} \leq K\|f\|_1 \;\forall\; f \in L_1(G)$.

Now consider the mapping $T_g : L_1(G) \to \mathbb{C}$ as follows :

$T_g(f) = f * g(0)$. Then by (5.4.1a) T_g is a bounded linear functional on $L_1(G)$. Hence $g \in (L_1(G))^* = L_\infty(G)$.

It is known [4] that if $A_p(G)$ $(1 \leq p < \infty)$ is dual of an L_1-module, then $M(L_1, A_p) = A_p$. In fact proof of this result in [4] gives the following: If $S(G)$ is dual of an L_1-module, then $M(L_1, S) = S(G)$. Using this fact we shall show that if $G = T$, the circle group, then $E_q(G)$ $(q > 1)$ is not dual of an L_1-module.

COROLLARY 5.4.2 : Let $G = T$, the circle group and $1 < q \leq 2$. Then $E_q(G)$ is not dual of an L_1-module.

PROOF : Suppose $E_q(G)$ is dual of an L_1-module. Then it will follow from Proposition 5.1.2 that $M(L_1, E_q) = E_q(G)$. Now from Proposition 5.4.1 we have

(5.4.2a) $\quad \{f \in L_\infty(G) \mid \hat{f} \in \ell_q(\Gamma)\} = E_q(G)$.

But (5.4.2a) is false, as can be seen from the following.

Let f be the function on G defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx .$$

It is clear that $\hat{f}(n) = \frac{1}{n} = o(1)$, hence $\hat{f} \in \ell_q(\Gamma)$ and $f \in L_\infty(G)$; however, $f \notin C(G)$ because $n\frac{1}{n} = 1$ does not tend to zero, (see [6, p 112]).

REMARK 5.4.3 : Because of Proposition 4.2.15 it can be easily seen that Proposition 5.4.1 is true for infinite compact groups.

5.5 MULTIPLIERS BETWEEN $A_p(G)$ and $E_q(G)$: It is easy to see that if $p \geq q$, then multipliers from $E_q(G)$ into $A_p(G)$ are precisely all bounded functions on Γ.

PROPOSITION 5.5.1 : If $p < q$, then

(i) $\ell_{pq/q-p}(\Gamma) \subseteq M_{E_q}^{A_p}(\Gamma) \subsetneq \ell_\infty(\Gamma)$

and $\ell_{pq/q-p}(\Gamma)$ is the largest $\ell_r(\Gamma)$ space contained in $M_{E_q}^{A_p}(\Gamma)$

(ii) $M_{E_2}^{A_p}(\Gamma) = \ell_{2p/2-p}(\Gamma)$, $p < 2$.

PROOF : By Proposition 4.2.8 $\ell_{pq/q-p}(\Gamma) = M_{A_q}^{A_p}(\Gamma) \subseteq M_{E_q}^{A_p}(\Gamma)$ and it is evident that $M_{E_q}^{A_p}(\Gamma) \subsetneq \ell_\infty(\Gamma)$. As in Proposition 4.2.9 it can be shown that $\ell_{pq/(q-p)}(\Gamma)$ is the largest $\ell_r(\Gamma)$ space contained in $M_{E_q}^{A_p}(\Gamma)$.

(ii) $\ell_{2p/2-p}(\Gamma) = M_{E_2}^{A_p}(\Gamma)$ follows from (i) and Theorem 2.4.8.

PROPOSITION 5.5.2 : Let $1 \leq p \leq 2$, then $M_{A_p}^{E_q}(\Gamma) = \ell_{p'}(\Gamma)$.

PROOF : It is clear that $\ell_{p'}(\Gamma) \subseteq M_{A_p}^{E_1}(\Gamma) \subseteq M_{A_p}^{E_q}(\Gamma)$. Conversely, since $(A_p(G))^\wedge = \ell_p(\Gamma)$, $M_{A_p}^{E_q}(\Gamma) \subseteq [\ell_p(\Gamma), L_\infty(G)] = \ell_{p'}(\Gamma)$.

Equality in last step follows from Theorem 2.4.5.

PROPOSITION 5.5.3 : Let $p > 2$, then

(i) $\ell_{p'}(\Gamma) \subsetneq M_{A_p}^{E_q}(\Gamma) \subsetneq (L_p(G))^{\wedge}$ ($q > 1$ for first inclusion to be proper).

PROOF : Evidently $\ell_{p'}(\Gamma) \subseteq M_{A_p}^{E_q}(\Gamma)$. Now to show that this inclusion is proper we observe that if $q > 1$, then $\frac{pq}{p-q} > \frac{p}{p-1} = p'$. By Proposition 5.1.3 choose a function $f \in E_{pq/p-q}(G)$ ($= C(G)$ if $\frac{pq}{p-q} \geq 2$) such that $\hat{f} \notin \ell_{p'}(\Gamma)$; then $\hat{f} \in M_{A_p}^{E_q}(\Gamma)$. Therefore

$$\ell_{p'}(\Gamma) \subsetneq M_{A_p}^{E_q}(\Gamma).$$

Now we shall show that $M_{A_p}^{E_q}(\Gamma) \subsetneq (L_p(G))^{\wedge}$. It is evident by Theorem 2.4.4 that

$$M_{A_p}^{E_q}(\Gamma) \subseteq [L_{p'}(G), C(G)] = (L_p(G))^{\wedge}. \tag{5.5.3a}$$

To show that inclusion in (5.5.3a) is proper, choose a function $f \in A_p(G)$ such that $f \notin L_{p'}(G)$; such a function exists because of [21]. Suppose $M_{A_p}^{E_q}(\Gamma) = (L_p(G))^{\wedge}$. Then $\hat{f}\,\hat{g} \in C(G)^{\wedge}$ for every $g \in L_p(G)$. That is, $\hat{f} \in [L_p, C] = L_{p'}(G)^{\wedge}$ by Theorem 2.4.4, a contradiction.

REMARK 5.5.4 : It is easy to see that results proved in this section are valid for infinite compact groups. The arguments used in proving these results are also valid for infinite compact groups. See Theorem 3.1.10.

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